Lehrstuhl A für Mathematik
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Aachen, den 14.07.2003

11. Übung zu Zahlbereichserweiterungen

(Abgabe: Montag, 21.07.2003, vor der Übung)

Aufgabe 1: Seien $g \in \mathbb{N}$, $g \geq 2$, $u, v \in \mathbb{N}$ mit $\operatorname{ggT}(u,v) = 1$ und

$$v = v_1 \cdot v_2, \quad v_1 = \prod_{p \in \mathbb{P},\, p \mid g} p^{\nu_p(v)}, \quad \operatorname{ggT}(v_2, g) = 1.$$

Dann gilt für die Periodenlänge τ und die Vorperiodenlänge σ einer g-adischen Bruchdarstellung von $\varepsilon\frac{u}{v}$:

$$\tau = \min\{k \in \mathbb{N};\ v_2 \,|\, (g^k - 1)\}, \qquad \sigma = \min\{l \in \mathbb{N}_0;\ v_1 \,|\, g^l\}.$$

Insbesondere hängen sie nur vom Nenner v ab. $\boxed{6}$

Aufgabe 2: Für $n \in \mathbb{N}$ sei $c_n := 1$, falls $n \in \mathbb{P}$ und $c_n := 0$, falls $n \notin \mathbb{P}$. Zeigen Sie, dass es kein $\gamma \in \mathbb{Q}$ gibt, sodass für irgendein $g \in \mathbb{N}$, $g \geq 2$ genau $\gamma = \sum_{k=1}^{\infty} c_k g^{-k}$ gilt. $\boxed{3}$

Aufgabe 3: Sei $g \in \mathbb{N}$, $g \geq 2$ und $(a_n)_{n \in \mathbb{N}}$ eine Abzählung von $\mathbb{Q}$, wobei die g-adische Darstellung von a_n gegeben sei durch $a_n = \varepsilon(a_n) \sum_{k=N(a_n)}^{\infty} a_{n_k} g^{-k}$. Zeigen Sie, dass dann $a := \sum_{k=1}^{\infty} a_{k_k} g^{-k} \notin \mathbb{Q}$. $\boxed{3}$

Aufgabe 4: Sei $n, g \in \mathbb{N}\backslash\{1\}$ mit $\operatorname{ggT}(n,g) = 1$. Der Stammbruch $\frac{1}{n}$ habe die Periodenlänge $n-1$ in der g-adischen Bruchdarstellung. Zeigen Sie, dass n eine Primzahl ist. $\boxed{3}$

Aufgabe 5:

a) Zeigen Sie $m^k - n^k > k$ für alle $m, n, k \in \mathbb{N}$ mit $m > n$ und $k > 1$. $\boxed{2}$

b) Es seien $x, y \in \mathbb{Q}$ mit $x > y > 0$. Zeigen Sie, dass genau dann $x^y = y^x$ gilt, wenn es ein $n \in \mathbb{N}$ gibt mit $x = \left(1 + \frac{1}{n}\right)^{n+1}$, $y = \left(1 + \frac{1}{n}\right)^n$.
Hinweis: Schreiben Sie $x = \frac{a}{b}$, $y = \frac{c}{d}$ mit $a, b, c, d \in \mathbb{N}$ und $\operatorname{ggT}(a,b) = \operatorname{ggT}(c,d) = 1$, und zeigen Sie $a^{bc} = c^{ad}$ und $b^{bc} = d^{ad}$. Finden Sie weiter $r, s, t \in\in \mathbb{N}$ mit $ad = rt$, $bc = st$, $\operatorname{ggT}(r,s) = 1$ und schließlich $m, n \in \mathbb{N}$ mit $a = m^r$, $b = n^r$, $c = m^s$, $d = n^s$. Beweisen Sie nun $r > s, t = m^s n^s$ und folgern Sie $r = m = n + 1$ sowie $s = n$. $\boxed{4}$

Aufgabe 6: Zeigen Sie, dass $\mathbb{C}$ wie in Satz 6.1 definiert ein Köper ist. Dabei können die aus der linearen Algebra bekannten Eigenschaften von Matrizen als bekannt vorausgesetzt werden. $\boxed{4}$

Aufgabe 7: Auf $\mathbb{C} \times \mathbb{C}$ definiert man das euklidische Skalarprodukt durch $< z, w >:= \mathrm{Re}(z\overline{w})$. Zeigen Sie für alle $w, z \in \mathbb{C}$:

a) Die Cauchy–Schwarzsche Ungleichung: $| < w, z > | \leq |w| \cdot |z|$. Dabei gilt das Gleichheitszeichen genau dann, wenn w und z linear abhängig über $\mathbb{R}$ sind. $\boxed{2}$

b) Den Cosinussatz: $|w + z|^2 = |w|^2 + |z|^2 + 2 < w, z >$. $\boxed{2}$

c) Die Dreiecksungleichung: $|w + z| \leq |w| + |z|$. $\boxed{2}$

$\boxed{31}$

Augustin Louis Cauchy[1]


(Teil 2) In 1815 Cauchy lost out to Binet for a mechanics chair at the Ecole Polytechnique, but then was appointed assistant professor of analysis there. He was responsible for the second year course. In 1816 he won the Grand Prix of the French Academy of Science for a work on waves. He achieved real fame however when he submitted a paper to the Institute solving one of Fermat's claims on polygonal numbers made to Mersenne. Politics now helped Cauchy into the Academy of Sciences when Carnot and Monge fell from political favour and were dismissed and Cauchy filled one of the two places.

In 1817 when Biot left Paris for an expedition to the Shetland Islands in Scotland Cauchy filled his post at the Collège de France. There he lectured on methods of integration which he had discovered, but not published, earlier. Cauchy was the first to make **a rigorous study of the conditions for convergence of infinite series** in addition to his rigorous definition of an integral. His text *Cours d'analyse* in 1821 was designed for students at Ecole Polytechnique and was concerned with developing the basic theorems of the calculus as rigorously as possible. He began a study of the calculus of residues in 1826 in *Sur un nouveau genre de calcul analogue au calcul infinétesimal* while in 1829 in *Lecons sur le Calcul Différential* he defined for the first time a **complex function of a complex variable**.

Cauchy did not have particularly good relations with other scientists. His staunchly Catholic views had him involved on the side of the Jesuits against the Académie des Sciences. He would bring religion into his scientific work as for example he did on giving a report on the theory of light in 1824 when he attacked the author for his view that Newton had not believed that people had souls.

Political events in France meant that Cauchy was now required to swear an oath of allegiance to the new regime and when he failed to return to Paris to do so he lost all his positions there. In 1831 Cauchy went to Turin and after some time there he accepted an offer from the King of Piedmont of a chair of theoretical physics. He taught in Turin from 1832.

In 1833 Cauchy went from Turin to Prague in order to follow Charles X and to tutor his grandson. However he was not very successful in teaching the prince as this description shows:

> When questioned by Cauchy on a problem in descriptive geometry, the prince was confused and hesitant. ... As with mathematics, the prince showed very little interest in these subjects. Cauchy became annoyed and screamed and yelled. The queen sometimes said to him, soothingly, smilingly, 'too loud, not so loud'.

In 1843 Lacroix died and Cauchy became a candidate for his mathematics chair at the Collège de France. Liouville and Libri were also candidates. Cauchy should have easily been appointed on his mathematical abilities but his political and religious activities, such as support for the Jesuits, became crucial factors. Libri was chosen, clearly by far the weakest of the three mathematically.

When Louis Philippe was overthrown in 1848 Cauchy regained his university positions. However he did not change his views and continued to give his colleagues problems. Libri, who had been appointed in the political way described above, resigned his chair and fled from France. Partly this must have been because he was about to be prosecuted for stealing valuable books. Liouville and Cauchy were candidates for the chair again in 1850 as they had been in 1843. After a close run election Liouville was appointed. Subsequent attempts to reverse this decision led to very bad relations between Liouville and Cauchy.

Another, rather silly, dispute this time with Duhamel clouded the last few years of Cauchy's life. This dispute was over a priority claim regarding a result on inelastic shocks. Duhamel argued with Cauchy's claim to have been the first to give the results in 1832. Poncelet referred to his own work of 1826 on the subject and Cauchy was shown to be wrong. However Cauchy was never one to admit he was wrong.

Numerous terms in mathematics bear Cauchy's name:- the *Cauchy integral theorem*, in the theory of complex functions, the *Cauchy-Kovalevskaya existence theorem* for the solution of partial differential equations, the *Cauchy-Riemann equations* and **Cauchy sequences**. He produced 789 mathematics papers, an incredible achievement.

[1] Aus: 'The MacTutor History of Mathematics archive' der University of St Andrews, Scotland.