Lehrstuhl A für Mathematik
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Aachen, den 1.12.1999

5. Übung zu Zahlbereichserweiterungen

(Abgabe: Mittwoch, 15.12.1999, bis 12 Uhr im Übungskasten)

Aufgabe 1: Zeigen Sie:

a) Für alle n, $N \in \mathbb{N}$ mit $n < (N+1)!$ existiert eine Darstellung

$$n = \sum_{k=1}^{N} a_k k!\,, \qquad a_k \in \{0, 1, \dots, k\}, \quad k = 1, \dots, N.$$

Kann man die a_k algorithmisch beschreiben? $\boxed{4}$

b) Jedes $n \in \mathbb{N}_0$ besitzt eine eindeutige Darstellung

$$n = \sum_{k=1}^{\infty} a_k k!\,, \qquad a_k \in \{0, 1, \dots, k\} \quad \text{für alle } k \in \mathbb{N}$$

und $a_k = 0$ für fast alle k. $\boxed{3}$

c) Wie groß muß man N in a) mindestens wählen um alle 100-stelligen natürlichen Zahlen darstellen zu können? $\boxed{2}$

Aufgabe 2: Ist $a = \sum_{i=0}^{n} q_i g^i$ die g-adische Darstellung von $a \in \mathbb{N}$, so heißt

$$Q'_q(a) := \sum_{i=0}^{n} (-1)^i q_i$$

alternierende g-adische Quersumme von a. Zeigen Sie:

$$(g+1)|a \Leftrightarrow (g+1)|Q'_q(a).$$

$\boxed{3}$

Aufgabe 3: Sei $m \in \mathbb{N}$ ungerade. Zeigen Sie die Äquivalenz der beiden Aussagen:

(i) m ist keine Primzahl.

(ii) Es gibt a, $b \in \mathbb{N}_0$ mit $a > b + 1$ und $m = a^2 - b^2$. $\boxed{4}$

Aufgabe 4: Bestimmen Sie die g-adische Darstellung der folgenden Zahlen:

a) 1993, $g = 12$, b) 553, $g = 5$, c) 422, $g = 8$. $\boxed{3}$

Aufgabe 5: Zeigen Sie, daß es zu jedem $a \in \mathbb{Q}$ genau ein $n \in \mathbb{Z}$ gibt mit $n \leq a < n+1$. Man nennt n das *größte Ganze* von a und schreibt $[a] := n$. $\boxed{3}$

Aufgabe 6: Sei $g \in \mathbb{N}$, $g \geq 2$ und

$$\mathbb{D} = \left\{ \varepsilon \sum_{k=-n}^{m} d_k g^k;\ \varepsilon = \pm 1,\ d_k \in \{0, \dots, g-1\},\ n,\ m \in \mathbb{N} \right\}.$$

Zeigen Sie:

a) $\mathbb{D} = \{\frac{m}{g^n};\ m \in \mathbb{Z},\ n \in \mathbb{N}_0\}$. $\boxed{3}$

b) $\mathbb{D}$ ist ein echter Unterring von $\mathbb{Q}$, der $\mathbb{Z}$ umfaßt. $\boxed{3}$

Johann Carl Friedrich Gauss[1]

Born: 30 April 1777 in Brunswick, Duchy of Brunswick (now Germany)
Died: 23 Feb 1855 in Gttingen, Hanover (now Germany)

At the age of seven, Carl Friedrich started elementary school, and his potential was noticed almost immediately. His teacher was amazed when Gauss summed the integers from 1 to 100 instantly by spotting that the sum was 50 pairs of numbers each pair summing to 101. In 1788 he began his education at the Gymnasium. After receiving a stipend from the Duke of Brunswick-Wolfenbttel, Gauss entered Brunswick Collegium Carolinum in 1792. At the academy Gauss independently discovered Bode's law, the binomial theorem and the arithmetic-geometric mean, as well as the law of quadratic reciprocity and the prime number theorem.
In 1795 Gauss left Brunswick to study at Gttingen University. Gauss's teacher there was Kaestner, whom Gauss often ridiculed. His only known friend amongst the students was Farkas Bolyai. They met in 1799 and corresponded with each other for many years. Gauss left Gttingen in 1798 without a diploma, but by this time he had made one of his most important discoveries - the construction of a regular 17-gon by ruler and compasses This was the most major advance in this field since the time of Greek mathematics and was published as Section VII of Gauss's famous work, *Disquisitiones Arithmeticae.*

[1] Aus: 'The MacTutor History of Mathematics archive' der University of St Andrews, Scotland.

Gauss returned to Brunswick where he received a degree in 1799. After the Duke of Brunswick had agreed to continue Gauss's stipend, he requested that Gauss submit a doctoral dissertation to the University of Helmstedt. He already knew Pfaff, who was chosen to be his advisor. Gauss's dissertation was a discussion of the fundamental theorem of algebra.

With his stipend to support him, Gauss did not need to find a job so devoted himself to research. He published the book *Disquisitiones Arithmeticae* in the summer of 1801. There were seven sections, all but the last section, referred to above, being devoted to number theory.

In June 1801, Zach, an astronomer whom Gauss had come to know two or three years previously, published the orbital positions of Ceres, a new "small planet" which was discovered by G. Piazzi, an Italian astronomer on 1 January, 1801. Unfortunately, Piazzi had only been able to observe 9 degrees of its orbit before it disappeared behind the Sun. Zach published several predictions of its position, including one by Gauss which differed greatly from the others. When Ceres was rediscovered by Zach on 7 December 1801 it was almost exactly where Gauss had predicted. Although he did not disclose his methods at the time, Gauss had used his least squares approximation method.

Gauss married Johanna Ostoff on 9 October, 1805. In 1807 he left Brunswick to take up the position of director of the Gttingen observatory. In 1808 his father died, and a year later Gauss's wife Johanna died after giving birth to their second son, who was to die soon after her.Gauss was married for a second time the next year, to Minna the best friend of Johanna, and although they had three children, this marriage seemed to be one of convenience for Gauss.

Gauss's work never seemed to suffer from his personal tragedy. He published his second book, *Theoria motus corporum coelestium in sectionibus conicis Solem ambientium,* in 1809, a major two volume treatise on the motion of celestial bodies.

Much of Gauss's time was spent on a new observatory, completed in 1816, but he still found the time to work on other subjects. His publications during this time include *Disquisitiones generales circa seriem infinitam*, a rigorous treatment of series and an introduction of the hypergeometric function, *Methodus nova integralium valores per approximationem inveniendi*, a practical essay on approximate integration, *Bestimmung der Genauigkeit der Beobachtungen*, a discussion of statistical estimators, and *Theoria attractionis corporum sphaeroidicorum ellipticorum homogeneorum methodus nova tractata*. The latter work was inspired by geodesic problems and was principally concerned with potential theory.

Gauss had been asked in 1818 to carry out a geodesic survey of the state of Hanover to link up with the existing Danish grid. Gauss was pleased to accept and took personal charge of the survey, making measurements during the day and reducing them at night, using his extraordinary mental capacity for calculations.

In 1822 Gauss won the Copenhagen University Prize with *Theoria attractionis ...* together with the idea of mapping one surface onto another so that the two *are similar in their smallest parts*. This paper was published in 1825 and led to the much later publication of *Untersuchungen ber Gegenstnde der Hheren Geodsie* (1843 and 1846).

From the early 1800's Gauss had an interest in the question of the possible existence of a non-Euclidean geometry. In a book review in 1816 he discussed proofs which deduced the axiom of parallels from the other Euclidean axioms, suggesting that he believed in the existence of non-Euclidean geometry, although he was rather vague.

Gauss had a major interest in differential geometry, and published many papers on the subject. *Disquisitiones generales circa superficies curva* (1828) was his most renowned work in this field.

In 1831, Wilhelm Weber arrived in Gttingen as physics professor filling Tobias Mayer's chair. Gauss had known Weber since 1828 and supported his appointment. Gauss had worked on physics before 1831, publishing *ber ein neues allgemeines Grundgesetz der Mechanik*, which contained the principle of least constraint, and *Principia generalia theoriae figurae fluidorum in statu aequilibrii* which discussed forces of attraction.
In 1832, Gauss and Weber began investigating the theory of terrestrial magnetism after Alexander von Humboldt attempted to obtain Gauss's assistance in making a grid of magnetic observation points around the Earth. Gauss was excited by this prospect and by 1840 he had written three important papers on the subject.These papers all dealt with the current theories on terrestrial magnetism, including Poisson's ideas, absolute measure for magnetic force and an empirical definition of terrestrial magnetism. Dirichlet's principal was mentioned without proof.
He showed that there can only be two poles in the globe and went on to prove an important theorem, which concerned the determination of the intensity of the horizontal component of the magnetic force along with the angle of inclination. Gauss used the Laplace equation to aid him with his calculations, and ended up specifying a location for the magnetic South pole.
Humboldt had devised a calendar for observations of magnetic declination. However, once Gauss's new magnetic observatory (completed in 1833 - free of all magnetic metals) had been built, he proceeded to alter many of Humboldt's procedures, not pleasing Humboldt greatly. However, Gauss's changes obtained more accurate results with less effort.
Gauss and Weber achieved much in their six years together. They discovered Kirchhoff's laws, as well as building a primitive telegraph device which could send messages over a distance of 5000 ft. However, this was just an enjoyable pastime for Gauss. He was more interested in the task of establishing a world-wide net of magnetic observation points. This occupation produced many concrete results. The *Magnetischer Verein* and its journal were founded, and the atlas of geomagnetism was published, while Gauss and Weber's own journal in which their results were published ran from 1836 to 1841.
Two of Gauss's last doctoral students were Moritz Cantor and Dedekind. Dedekind wrote a fine description of his supervisor

> *... usually he sat in a comfortable attitude, looking down, slightly stooped, with hands folded above his lap. He spoke quite freely, very clearly, simply and plainly: but when he wanted to emphasise a new viewpoint ... then he lifted his head, turned to one of those sitting next to him, and gazed at him with his beautiful, penetrating blue eyes during the emphatic speech. ... If he proceeded from an explanation of principles to the development of mathematical formulas, then he got up, and in a stately very upright posture he wrote on a blackboard beside him in his peculiarly beautiful handwriting: he always succeeded through economy and deliberate arrangement in making do with a rather small space. For numerical examples, on whose careful completion he placed special value, he brought along the requisite data on little slips of paper.*

Gauss died in his sleep early in the morning of 23 February, 1855.
Carl F Gauss was elected to the Royal Society of London in 1804. He was awarded the Copley Medal of the Royal Society in 1838. There is a Crater Gauss on the moon.