LEHRSTUHL A FÜR MATHEMATIK
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Aachen, den 17.11.1999

4. Übung zu Zahlbereichserweiterungen

(Abgabe: Mittwoch, 1.12.1999, bis 12 Uhr im Übungskasten)

Definition : Seien $a_1, ..., a_n \in \mathbb{Z}$. Dann nennt man $d \in \mathbb{N}_0$ ein *kleinstes gemeinsames Vielfaches* von $a_1, ..., a_n$ und schreibt $d = \mathrm{kgV}(a_1, ..., a_n)$, wenn $a_i | d$, $i = 1, ..., n$ und aus $a_i | d'$, $i = 1, ..., n$ bereits $d | d'$ folgt.

Aufgabe 1: Beweisen Sie den *Fundamentalsatz der Arithmetik:*
Jede natürliche Zahl $n > 1$ läßt sich als endliches Produkt von Primzahlpotenzen darstellen, und die Darstellung ist bis auf die Reihenfolge eindeutig. Es gibt ein $r \in \mathbb{N}$ sowie $p_1, ..., p_r \in \mathbb{P}$ und $\nu_1, ..., \nu_r \in \mathbb{N}$ mit

$$n = p_1^{\nu_1} \cdot ... \cdot p_r^{\nu_r}.$$

Durch die Bedingung $1 < p_1 < ... < p_r$ wird die Darstellung eindeutig bestimmt. $\boxed{4}$

Aufgabe 2: Seien $a,\ m,\ n \in \mathbb{N}$, $a \geq 2$. Zeigen Sie:

a) $a^m - 1$ ist stets ein Teiler von $a^{mn} - 1$. $\boxed{2}$

b) Ist $m > 1$ und $a^m - 1$ eine Primzahl, so ist m eine Primzahl und $a = 2$. $\boxed{3}$

c) Bestimmen Sie den größten gemeinsamen Teiler von $a^m - 1$ und $a^n - 1$. $\boxed{3}$

Aufgabe 3: Zeigen Sie:

a) Sind $a_1, ..., a_n \in \mathbb{Z}$, so existiert $\mathrm{kgV}(a_1, ..., a_n)$ und ist eindeutig bestimmt. $\boxed{2}$

b) $\mathrm{ggT}(a, b) \cdot \mathrm{kgV}(a, b) = |ab|$ für $a,\ b \in \mathbb{Z}$. $\boxed{2}$

c) $\mathbb{Z}a_1 \cap \ldots \cap \mathbb{Z}a_n = \mathbb{Z} \cdot \mathrm{kgV}(a_1, ..., a_n)$ für $n \in \mathbb{N}$, $a_i \in \mathbb{Z}$, $i = 1\ \ldots, n$. $\boxed{3}$

d) Sind $a,\ b,\ c,\ n \in \mathbb{N}$ mit $ab = c^n$ und $\mathrm{ggT}(a, b) = 1$, so existieren $\alpha,\ \beta \in \mathbb{N}$ mit $a = \alpha^n$ und $b = \beta^n$. $\boxed{3}$

e) Für $a_1, \ldots, a_n,\ b_1, \ldots, b_n \in \mathbb{Z}$ mit $|a_i b_i| = c$ für $i = 1, \ldots, n$ gilt:

$$c = \mathrm{ggT}(a_1, ..., a_n) \cdot \mathrm{kgV}(b_1, ..., b_n)$$

$\boxed{3}$

Aufgabe 4: Bestimmen Sie den größten gemeinsamen Teiler und das kleinste gemeinsame Vielfache der Zahlen 1819 und 3587. Stellen Sie den größten gemeinsamen Teiler als ganzzahlige Linearkombination dar. $\boxed{4}$

Aufgabe 5: Seien $a,\ b \in \mathbb{N}$ teilerfremd und $a \geq b$. Zeigen Sie, daß es eindeutig bestimmte $m,\ n \in \mathbb{N}_0$ gibt mit den Eigenschaften:

$$ma - nb = 1, \quad m \leq b,\ n < a.$$

$\boxed{5}$

Giuseppe Peano[1]

Born: 27 Aug 1858 in Cuneo, Piemonte, Italy
Died: 20 April 1932 in Turin, Italy

Giuseppe Peano's parents worked on a farm and Giuseppe was born in the farmhouse 'Tetto Galant' about 5 km from Cuneo. He attended the village school in Spinetta then he moved up to the school in Cuneo, making the 5km journey there and back on foot every day.

Giuseppe's mother had a brother who was a priest and lawyer in Turin and, when he realised that Giuseppe was a very talented child, he took him to Turin in 1870 for his secondary schooling and to prepare him for university studies. Giuseppe entered the University of Turin in 1876. On 29 September 1880 he graduated as doctor of mathematics. and joined the staff at the University of Turin. He received his qualification to be a university professor in December 1884.

In 1886 Peano proved that if $f(x, y)$ is continuous then the first order differential equation $dy/dx = f(x, y)$ has a solution. In 1888 he published the book *Geometrical Calculus* which begins with a chapter on mathematical logic. This was his first work on the topic that would play a major role in his research over the next few years and it was based on the work of Schrder, Boole and Charles Peirce. This book contains the first definition of a vector space given with a remarkably modern notation and style and, although it was not appreciated by many at the time, this is surely a quite remarkable achievement by Peano.

In 1889 he published his famous axioms, called Peano axioms, which defined the natural numbers in terms of sets. These were published in a pamphlet *Arithmetices principia, nova methodo exposita* which were a landmark in the history of mathematical logic and of the foundations of mathematics.

He invented 'space-filling' curves in 1890, these are continuous surjective mappings from $[0, 1]$ onto the unit square. Hilbert, in 1891, described similar space-filling curves. It had been thought that such curves could not exist. Hausdorff wrote of Peano's result in *Grundzge der Mengenlehre* in 1914: *This is one of the most remarkable facts of set theory.*

From around 1892, Peano embarked on a new and extremely ambitious project, namely the *Formulario Mathematico.* He explained in the March 1892 part of *Rivista di matematica* his thinking:

> *Of the greatest usefulness would be the publication of collections of all the theorems now known that refer to given branches of the mathematical sciences ... Such a collection, which would be long and difficult in ordinary language, is made noticeably easier by using the notation of mathematical logic ...*

The *Formulario Mathematico* project was completed in 1908 and one has to admire what Peano achieved but although the work contained a mine of information it was little used. Although Peano is a founder of mathematical logic, the German mathematical philosopher Gottlob Frege is today considered the father of mathematical logic.

[1] Aus: 'The MacTutor History of Mathematics archive' der University of St Andrews, Scotland.