Lehrstuhl A für Mathematik
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Aachen, den 17.01.2002

11. Übung zu Zahlbereichserweiterungen

(Abgabe: Donnerstag, 24.01.2002, vor der Übung oder
bis 10 Uhr im Übungskasten vor dem Sekretariat des Lehrstuhls)

Definition. Ein Tripel $(a, b, c) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ heißt ein *Pythagoräisches Tripel*, wenn $a^2 + b^2 = c^2$ gilt. Man nennt es darüber hinaus *primitiv*, wenn $\mathrm{ggT}(a, b, c) = 1$ gilt.

Aufgabe 1: Sei (a, b, c) ein Pythagoräisches Zahlentripel. Zeigen Sie:

a) 4 ist ein Teiler von a oder von b. $\boxed{3}$

b) 3 ist ein Teiler von a oder von b. $\boxed{3}$

c) 5 ist ein Teiler von a, b oder c. $\boxed{3}$

Aufgabe 2: Zeigen Sie: Die Gleichung $a^4 + b^2 = c^4$ besitzt keine Lösung $(a, b, c) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$. $\boxed{5}$

Aufgabe 3:

a) Zeigen Sie $m^k - n^k > k$ für alle $m, n, k \in \mathbb{N}$ mit $m > n$ und $k > 1$. $\boxed{2}$

b) Es seien $x, y \in \mathbb{Q}$ mit $x > y > 0$. Zeigen Sie, dass genau dann $x^y = y^x$ gilt, wenn es ein $n \in \mathbb{N}$ gibt mit $x = \left(1 + \frac{1}{n}\right)^{n+1}$, $y = \left(1 + \frac{1}{n}\right)^{n+1}$.
Hinweis: Schreiben Sie $x = \frac{a}{b}$, $y = \frac{c}{d}$ mit $a, b, c, d \in \mathbb{N}$ und $ggT(a, b) = ggT(c, d) = 1$, und zeigen Sie $a^{bc} = c^{ad}$ und $b^{bc} = d^{ad}$. Finden Sie weiter $r, s, t \in\in \mathbb{N}$ mit $ad = rt$, $bc = st$, $ggT(r, s) = 1$ und schließlich $m, n \in \mathbb{N}$ mit $a = m^r$, $b = n^r$, $c = m^s$, $d = n^s$. Beweisen Sie nun $r > s, t = m^s n^s$ und folgern Sie $r = m = n + 1$ sowie $s = n$. $\boxed{4}$

Aufgabe 4: Zeigen Sie, dass $\mathbb{C}$ wie in Satz 6.1 definiert ein Körper ist. Dabei können die aus der linearen Algebra bekannten Eigenschaften von Matrizen als bekannt vorausgesetzt werden. $\boxed{4}$

Aufgabe 5: Auf $\mathbb{C} \times \mathbb{C}$ definiert man das euklidische Skalarprodukt durch $< z, w >:= \mathrm{Re}(z\overline{w})$. Zeigen Sie für alle $w, z \in \mathbb{C}$:

a) Die Cauchy–Schwarzsche Ungleichung: $| < w, z > | \leq |w| \cdot |z|$. Dabei gilt das Gleichheitszeichen genau dann, wenn w und z linear abhängig über $\mathbb{R}$ sind. $\boxed{2}$

b) Den Cosinussatz: $|w + z|^2 = |w|^2 + |z|^2 + 2 < w, z >$. $\boxed{2}$

c) Die Dreiecksungleichung: $|w + z| \leq |w| + |z|$. $\boxed{2}$

$\boxed{30}$

Johann Carl Friedrich Gauss[1]

Born: 30 April 1777 in Brunswick, Duchy of Brunswick (now Germany)
Died: 23 Feb 1855 in Göttingen, Hanover (now Germany)

At the age of seven, Carl Friedrich started elementary school, and his potential was noticed almost immediately. His teacher was amazed when Gauss summed the integers from 1 to 100 instantly by spotting that the sum was 50 pairs of numbers each pair summing to 101. In 1788 he began his education at the Gymnasium. After receiving a stipend from the Duke of Brunswick-Wolfenbüttel, Gauss entered Brunswick Collegium Carolinum in 1792. At the academy Gauss independently discovered Bode's law, the binomial theorem and the arithmetic-geometric mean, as well as the law of quadratic reciprocity and the prime number theorem.

In 1795 Gauss left Brunswick to study at Göttingen University. Gauss's teacher there was Kaestner, whom Gauss often ridiculed. His only known friend amongst the students was Farkas Bolyai. They met in 1799 and corresponded with each other for many years.

Gauss left Göttingen in 1798 without a diploma, but by this time he had made one of his most important discoveries - the construction of a regular 17-gon by ruler and compasses This was the most major advance in this field since the time of Greek mathematics and was published as Section VII of Gauss's famous work, *Disquisitiones Arithmeticae.*

Gauss returned to Brunswick where he received a degree in 1799. After the Duke of Brunswick had agreed to continue Gauss's stipend, he requested that Gauss submit a doctoral dissertation to the University of Helmstedt. He already knew Pfaff, who was chosen to be his advisor. Gauss's dissertation was a discussion of the **fundamental theorem of algebra.**

With his stipend to support him, Gauss did not need to find a job so devoted himself to research. He published the book *Disquisitiones Arithmeticae* in the summer of 1801. There were seven sections, all but the last section, referred to above, being devoted to number theory.

In June 1801, Zach, an astronomer whom Gauss had come to know two or three years previously, published the orbital positions of Ceres, a new "small planet"which was discovered by G. Piazzi, an Italian astronomer on 1 January, 1801. Unfortunately, Piazzi had only been able to observe 9 degrees of its orbit before it disappeared behind the Sun. Zach published several predictions of its position, including one by Gauss which differed greatly from the others. When Ceres was rediscovered by Zach on 7 December 1801 it was almost exactly where Gauss had predicted. Although he did not disclose his methods at the time, Gauss had used his least squares approximation method. (Fortsetzung folgt)

[1] Aus: 'The MacTutor History of Mathematics archive' der University of St Andrews, Scotland.