Lehrstuhl A für Mathematik
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6. Übung zu Zahlbereichserweiterungen

(Abgabe: Donnerstag, 06.12.2001, vor der Übung oder
bis 10 Uhr im Übungskasten vor dem Sekretariat des Lehrstuhls)

Aufgabe 1: Sei $m \in \mathbb{N}, m > 1$ ungerade. Zeigen Sie die Äquivalenz der beiden Aussagen:

a) m ist keine Primzahl.

b) Es gibt $a, b \in \mathbb{N}_0$ mit $a > b + 1$ und $m = a^2 - b^2$. $\boxed{3}$

Aufgabe 2: Sei $(R, <)$ ein angeordneter, unitärer Ring. Beweisen Sie, dass für alle $a, b \in R$ gilt:

a) $a \leq |a|$, $|a| \geq 0$ und $|a| = 0 \iff a = 0$. $\boxed{2}$

b) $|-a| = |a|$. $\boxed{2}$

c) $|ab| = |a|\,|b|$. $\boxed{2}$

d) $|a^{-1}| = |a|^{-1}$, falls $a^{-1} \in R$ existiert. $\boxed{2}$

e) $|a + b| \leq |a| + |b|$ (Dreiecksungleichung). $\boxed{2}$

f) $|a - b| \geq \big||a| - |b|\big|$. $\boxed{2}$

Aufgabe 3: Seien $a, m, n \in \mathbb{N}, a \geq 2$. Zeigen Sie:

a) $a^m - 1$ ist stets ein Teiler von $a^{mn} - 1$. $\boxed{2}$

b) Ist $m > 1$ und $a^m - 1$ eine Primzahl, so ist m eine Primzahl und $a = 2$. $\boxed{3}$

c) Bestimmen Sie den größten gemeinsamen Teiler von $a^m - 1$ und $a^n - 1$. $\boxed{3}$

Definition : *Seien $a_1, \ldots, a_n \in \mathbb{Z}$. Dann nennt man $d \in \mathbb{N}_0$ ein kleinstes gemeinsames Vielfaches von $a_1, \ldots, a_n$ und schreibt $d = kgV(a_1, \ldots, a_n)$, wenn $a_i \mid d, i = 1, \ldots, n$ und aus $a_i \mid d', i = 1, \ldots, n$ bereits $d \mid d'$ folgt.*

Aufgabe 4: Zeigen Sie:

a) Sind $a_1, \ldots, a_n \in \mathbb{Z}$, so existiert $kgV(a_1, \ldots, a_n)$ und ist eindeutig bestimmt. $\boxed{2}$

b) $ggT(a, b) \cdot kgV(a, b) = |ab|$ für $a, b \in \mathbb{Z}$. $\boxed{2}$

c) $\mathbb{Z}a_1 \cap \cdots \cap \mathbb{Z}a_n = \mathbb{Z} \cdot kgV(a_1, \ldots, a_n)$ für $n \in \mathbb{N}, a_i \in \mathbb{Z}, i = 1, \ldots, n$. $\boxed{3}$

d) Sind $a, b, c, n \in \mathbb{N}$ mit $ab = c^n$ und $ggT(a, b) = 1$, so existieren $\alpha, \beta \in \mathbb{N}$ mit $a = \alpha^n$ und $b = \beta^n$. $\boxed{3}$

e) Für $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{Z}$ mit $|a_i b_i| = c$ für $i = 1, \ldots, n$ gilt:

$$c = ggT(a_1, \ldots, a_n) \cdot kgV(b_1, \ldots, b_n).$$ $\boxed{3}$

Aufgabe 5: Bestimmen Sie den größten gemeinsamen Teiler und das kleinste gemeinsame Vielfache der Zahlen 1819 und 3587. Stellen Sie den größten gemeinsamen Teiler als ganzzahlige Linearkombination dieser Zahlen dar. $\boxed{4}$

$\boxed{\boxed{40}}$

Euclid of Alexandria[1]

Born: about 325 BC
Died: about 265 BC in Alexandria, Egypt

Euclid of Alexandria is the most prominent mathematician of antiquity best known for his treatise on mathematics *The Elements*. The long lasting nature of *The Elements* must make Euclid the leading mathematics teacher of all time. However little is known of Euclid's life except that he taught at Alexandria in Egypt.

The Elements was a compilation of knowledge that became the centre of mathematical teaching for 2000 years. Probably no results in *The Elements* were first proved by Euclid but the organisation of the material and its exposition are certainly due to him. In fact there is ample evidence that Euclid is using earlier textbooks as he writes the *Elements* since he introduces quite a number of definitions which are never used such as that of an oblong, a rhombus, and a rhomboid.

The Elements is divided into 13 books. Books one to six deal with plane geometry. In particular books one and two set out basic properties of triangles, parallels, parallelograms, rectangles and squares. Book three studies properties of the circle while book four deals with problems about circles and is thought largely to set out work of the followers of Pythagoras. Book five lays out the work of Eudoxus on proportion applied to commensurable and incommensurable magnitudes. Book six looks at applications of the results of book five to plane geometry.

Books seven to nine deal with number theory. In particular **book seven** is a self-contained introduction to number theory and **contains the Euclidean algorithm for finding the greatest common divisor of two numbers**. Book eight looks at numbers in geometrical progression.

Book ten deals with the theory of irrational numbers and is mainly the work of Theaetetus. Euclid changed the proofs of several theorems in this book so that they fitted the new definition of proportion given by Eudoxus.

Books eleven to thirteen deal with three-dimensional geometry. In book thirteen the basic definitions needed for the three books together are given. The theorems then follow a fairly similar pattern to the two-dimensional analogues previously given in books one and four. The main results of book twelve are that circles are to one another as the squares of their diameters and that spheres are to each other as the cubes of their diameters. These results are certainly due to Eudoxus. Euclid proves these theorems using the *method of exhaustion* as invented by Eudoxus. The Elements ends with book thirteen which discusses the properties of the five regular polyhedra and gives a proof that there are precisely five. This book appears to be based largely on an earlier treatise by Theaetetus.

Euclid's *Elements* is remarkable for the clarity with which the theorems are stated and proved. The standard of rigour was to become a goal for the inventors of the calculus centuries later. More than one thousand editions of *The Elements* have been published since it was first printed in 1482. B L van der Waerden assesses the importance of the *Elements*:

> *Almost from the time of its writing and lasting almost to the present, the Elements has exerted a continuous and major influence on human affairs. It was the primary source of geometric reasoning, theorems, and methods at least until the advent of non-Euclidean geometry in the 19th century. It is sometimes said that, next to the Bible, the* Elements *may be the most translated, published, and studied of all the books produced in the Western world.*

Euclid also wrote the following books which have survived: *Data* (with 94 propositions), which looks at what properties of figures can be deduced when other properties are given; *On Divisions* which looks at constructions to divide a figure into two parts with areas of given ratio; *Optics* which is the first Greek work on perspective; and *Phaenomena* which is an elementary introduction to mathematical astronomy and gives results on the times stars in certain positions will rise and set.

Euclid may not have been a first class mathematician but the long lasting nature of The Elements must make him the leading mathematics teacher of antiquity or perhaps of all time.

[1] Aus: 'The MacTutor History of Mathematics archive' der University of St Andrews, Scotland.