Lehrstuhl A für Mathematik
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7. Übung zur Vorlesung Topologie

(Abgabe: Freitag, 09.06.2000, bis 11.45 Uhr im Übungskasten)

Aufgabe 1: a) Sei $X := \mathbb{R}$. Bestimmen Sie die bezüglich der Menge aller monoton wachsenden Funktionen $f : X \to (\mathbb{R}, \mathcal{T}_{nat})$ initiale Topologie auf X. $\boxed{3}$
b) Sei $\mathbb{K}$ ein Körper. Wie läßt sich die Zariskitopologie auf $\mathbb{K}^n$ als Initialtopologie darstellen? $\boxed{3}$

Aufgabe 2: Sei $(X, \mathcal{T})$ ein topologischer Raum mit $X \neq \emptyset$. Sei $\mathcal{T}^*$ die Topologie auf $\mathbb{R}^*$ aus Aufgabe 2, Blatt 2. Zeigen Sie:

a) Eine Abbildung $f : \mathbb{R}^* \to \mathbb{R}$ ist genau dann stetig, wenn $f|_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ stetig ist und die beiden Limiten $\lim\limits_{x\to+\infty} f|_{\mathbb{R}}(x)$ bzw. $\lim\limits_{x\to-\infty} f|_{\mathbb{R}}(x)$ existieren und mit $f(+\infty)$ bzw. $f(-\infty)$ übereinstimmen. $\boxed{3}$

b) Eine Abbildung $f : X \to \mathbb{R}^*$ ist genau dann stetig, wenn für jedes $c \in \mathbb{R}$ die Mengen $\{x \in X; f(x) > c\}$ und $\{x \in X; f(x) < c\}$ offen sind. $\boxed{2}$

c) Sind $f_j : X \to \mathbb{R}^*$, $j = 1, \ldots, n$, stetig, so sind auch die durch

$$M(x) := \max_{1\le j\le n} f_j(x) \text{ und } m(x) := \min_{1\le j\le n} f_j(x)$$

definierten Abbildungen $M : X \to \mathbb{R}^*$ und $m : X \to \mathbb{R}^*$ stetig. $\boxed{3}$

d) Für $n \in \mathbb{Z}$, $n \geq 0$, definiere man $g_n : \mathbb{R} \to \mathbb{R}^*$ durch $g_n(x) = x^{-n}$ für $x \neq 0$ und $g_n(0) := +\infty$. Welche der Abbildungen g_n sind stetig? $\boxed{3}$

Aufgabe 3:

(i) Seien $X := [0, 2\pi]$ und $Y := [-1, 1]$ versehen mit der von der Abstandsmetrik erzeugten Topologie. Zeigen Sie:
$f : X \to Y$, $f(x) := \sin x$, ist abgeschlossen, aber nicht offen. $\boxed{3}$

(ii) Sei $P_1 \in \mathbb{R}[X]$ ein Polynom mit reellen Koeffizienten. Zeigen Sie:
Die Polynomfunktion $p_1 : \mathbb{R} \to \mathbb{R}$, $p_1(x) := P_1(x)$, ist abgeschlossen, aber nicht notwendigerweise offen bezüglich der natürlichen Topologie auf $\mathbb{R}$. $\boxed{3}$

(iii) Sei $P_2 \in \mathbb{C}[X]$ ein nicht–konstantes Polynom mit komplexen Koeffizienten. Zeigen Sie:
Die Polynomfunktion $p_2 : \mathbb{C} \to \mathbb{C}$, $p_2(z) := P_2(z)$ ist offen und abgeschlossen bezüglich der natürlichen Topologie auf $\mathbb{C}$. $\boxed{3}$

Aufgabe 4: Seien $(Y, \mathcal{T}_{rel})$ ein topologischer Unterraum von $(X, \mathcal{T}), A, V \subset Y$ und $y \in Y$. Zeigen Sie:

a) V ist genau dann Umgebung von y bzgl. $\mathcal{T}_{rel}$, wenn eine Umgebung U von y bzgl. $\mathcal{T}$ existiert mit $V = U \cap Y$, d.h. $\mathfrak{U}_{rel}(y) = \{U \cap Y \subset Y;\ U \in \mathfrak{U}(y)\}$. $\boxed{2}$

b) $(A^+)_{\mathcal{T}_{rel}} = (A^+)_{\mathcal{T}} \cap Y$. $\boxed{2}$

c) $\mathring{A} \subset B$, wobei B der offene Kern von A in der Relativtopologie auf Y ist. $\boxed{2}$

d) $C \subset Y \cap \partial A$, wobei C der Rand von A in der Relativtopologie auf Y ist. $\boxed{2}$

e) Zeigen Sie an Beispielen, dass weder in c) noch in d) Gleichheit gelten muss. $\boxed{2}$

f) Ist $\mathfrak{B}$ eine Basis von $\mathcal{T}$, dann ist $\mathfrak{B}_{rel} := \{B \cap Y \subset Y;\ B \in \mathfrak{B}\}$ eine Basis von $\mathcal{T}_{rel}$. $\boxed{2}$

g) Ist $\mathfrak{B}(y)$ eine Umgebungsbasis von $y \in Y$ bzgl. $\mathcal{T}$, dann ist $\mathfrak{B}_{rel}(y) := \{B \cap Y \subset Y;\ B \in \mathfrak{B}(y)\}$ eine Umgebungsbasis von y bzgl.$\mathcal{T}_{rel}$. $\boxed{2}$

$\boxed{\boxed{40}}$

Die Übungsblätter sind auch im Internet erhältlich unter

http://www.mathA.rwth-aachen.de/lehre/SS00/Topologie

Oscar Zariski[1]

Born: 24 April 1899 in Kobrin, Belarus, Russian Empire
Died: 4 July 1986 in Brookline, Massachusetts, USA


Oscar Zariski's father was Bezalel Zaritsky and his mother Hannah Zaritsky. Oscar, born into this Jewish family, was named Ascher Zaritsky by his parents. We will comment below on how his name came to be changed to the now familiar version of Oscar Zariski.

Oscar's mother was the one who ensured that her young son had a good education. A tutor was provided for Oscar from the time he was seven years old and Oscar, under the guidance of the tutor, showed remarkable aptitude for the Russian language and for arithmetic. When the fighting associated with World War I reached Belarus, Oscar's family fled to Chernigov in the Ukraine.

Unable to enter the faculty of mathematics at the University of Kiev as all the places were full, he chose philosophy instead. He was a student of philosophy at Kiev from 1918 to 1920. However he was able to pursue his mathematical interests and studied algebra and number theory in addition to philosophy.

In 1920 he decided to go to Italy to continue his studies. In Rome Zariski came under the influence of the great algebraic geometers Castelnuovo, Enriques and Severi. He obtained a doctorate from Rome in 1924 for a doctoral thesis on a topic related to Galois theory which was proposed to him by Castelnuovo. Zariski married in 1924; having met his future wife Yole Cagli in Rome they returned to Zariski's home town of Kobin to marry. Returning to Rome he remained there as a fellow of the International Education Board until 1927.

It was while Zariski was in Rome that Enriques suggested that Ascher Zaritsky, as he was then called, change his name to the Italian sounding Oscar Zariski.

Now Zariski had gone to Italy to escape the problems in Belarus and the Ukraine. However, the political situation in Italy began to deteriorate rapidly. In October 1922 Mussolini organized the Fascist "March on Romeänd he was asked to form a government. The Fascist hatred of Jews made life for Zariski, because of his Jewish background, particularly difficult. Helped by Lefschetz, he escaped from the political problems of Italy in 1927 and went to the United States. He became a full professor at Johns Hopkins University in 1937.

[1]Aus: 'The MacTutor History of Mathematics archive' der University of St Andrews, Scotland. (gekrzt)

Now Castelnuovo and Severi had encouraged Zariski to view Lefschetz's topological methods as being the road ahead for algebraic geometry, so between 1927 and 1937 Zariski frequently visited Lefschetz at Princeton.

During this period Zariski wrote *Algebraic Surfaces* which was published in 1935. Between 1939 and 1940 he carried out his project of applying modern algebra to the foundations of algebraic geometry.

An important year for Zariski was 1945 which he spent in Sao Paolo. There he gave a lecture course three days each week which was attended by Andr Weil and nobody else. Both Zariski and Weil learnt much in discussions, often arguments, about the material that Zariski was presenting. After spending the year 1946–47 at the University of Illinois, Zariski was appointed to a chair at Harvard where he was to remain until he retired in 1969. From the late 1970s he suffered from Alzheimer's disease and his last few years were difficult ones as his health failed.

In 1981 Zariski was awarded the Steele Prize by the American Mathematical Society for the cumulative influence of his total mathematical research. The citation for the prize summarised Zariski's contributions to mathematics throughout his life:

... *All of Zariski's work has served as a basis for the present flowering of algebraic geometry and the current school uses his work and ideas in the modern development of the subject.*

Zariski's most famous book is *Commutative Algebra*, a two volume work written jointly with P Samuel. The first volume appeared in 1958, the second in 1960.

The American Mathematical Society played a large role in Zariski's life and he contributed greatly to the Society over many years. He was vice president of the Society between 1960 and 1961 and president of the Society from 1969 to 1970.

Zariski played an important role in mathematical publishing after his appointment as a full professor. He was an editor of the *American Mathematical Journal* from 1937 to 1941, served as a member of the editorial committee of the *Transactions of the American Mathematical Society* from 1941 to 1947, and also served on the editorial boards of the *Annals of Mathematics* and the *American Journal of Mathematics*.

After going to the United States in 1927 Zariski spent considerable periods lecturing at other universities both in the United States and in other countries.

He was awarded many honours for his work in addition to the Steele Prize described above. He was awarded the Cole Prize in Algebra from the American Mathematical Society in 1944 for four papers on algebraic varieties. He was awarded the National Medal of Science in 1965. Many academies and societies have honoured him by electing him to membership.