Lehrstuhl A für Mathematik
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Aachen, den 28.04.2000

2. Übung zur Vorlesung Topologie

(Abgabe: Freitag, 05.05.2000, bis 11.45 Uhr im Übungskasten)

Aufgabe 1: Sei X eine Menge. Eine Teilmenge $\mathcal{P} \neq \emptyset$ der Potenzmenge $\mathcal{P}(X)$ heißt eine *Partition* von X, wenn je zwei Elemente von $\mathcal{P}$ disjunkt sind und $\cup_{P \in \mathcal{P}} P = X$ gilt.

a) Sei $\mathcal{P}$ eine Partition von X. Zeigen Sie, daß $\mathcal{P}$ die Basis einer Topologie $\mathcal{T}_{\mathcal{P}}$ auf X ist. Diese Topologie heißt *Partitionstopologie* (zur Partition $\mathcal{P}$). Welche Partitionen muß man betrachten, um die diskrete bzw. indiskrete Topologie auf X zu erhalten? $\boxed{2}$

b) Sei $\mathcal{T}$ eine Topologie auf X. Zeigen Sie die Äquivalenz der beiden folgenden Aussagen:

(i) $\mathcal{T}$ stimmt mit der Menge der abgeschlossenen Teilmengen von X überein.

(ii) Es gibt eine Partition $\mathcal{P}$ von X mit $\mathcal{T} = \mathcal{T}_{\mathcal{P}}$. $\boxed{3}$

Aufgabe 2:

a) Sei $\mathcal{T}_n$ die natürliche Topologie auf $\mathbb{R}$, $\mathbb{R}^* = \mathbb{R} \cup \{+\infty, -\infty\}$ und

$$\mathcal{B} = \mathcal{T}_n \cup \{\,]a, +\infty[\cup \{+\infty\} \; ; \; a \in \mathbb{R}\} \cup \{\,]-\infty, b[\cup \{-\infty\} \; ; \; b \in \mathbb{R}\}.$$

Zeigen Sie:
$\mathcal{B}$ ist Basis einer Topologie $\mathcal{T}^*$ auf $\mathbb{R}^*$. $\boxed{3}$

b) Im $\mathbb{R}^2$ definiert man die obere Halbebene durch

$$H := \{(x, y) \in \mathbb{R}^2 \; ; \; y \geq 0\}.$$

Für $r > 0$ und $(x_0, y_0) \in \mathbb{R}^2$ ist

$$K_r(x_0, y_0) := \{(x, y) \in \mathbb{R}^2 \; ; \; (x - x_0)^2 + (y - y_0)^2 < r^2\}$$

die offene Kreisscheibe mit Mittelpunkt (x_0, y_0) und Radius r.

Zeigen Sie, daß

$$\mathcal{B} := \{K_r(x, y) \; ; \; (x, y) \in H, 0 < r \leq y\} \cup \{K_r(x, r) \cup \{(x, 0)\} \; ; \; x \in \mathbb{R}, r > 0\}$$

Basis einer Topologie $\mathcal{T}$ auf H ist. $\boxed{3}$

Definition. Sei $(X, \mathcal{T})$ ein topologischer Raum. Dann definiert man den *(topologischen) Abschluss* bzw. die *abgeschlossene Hülle* $\overline{A}$ einer Menge $A \subset X$ durch $\overline{A} := A \cup A^+$.

Aufgabe 3: Sei $(X, \mathcal{T})$ ein topologischer Raum und $A, B \subset X$. Zeigen Sie:

a) $f \in A^+ \Rightarrow f \in (A \backslash \{f\})^+$. $\boxed{2}$

b) $(A \cup B)^+ = A^+ \cup B^+$. $\boxed{3}$

c) $\overline{A \cup B} = \overline{A} \cup \overline{B}$. $\boxed{2}$

d) $f \in \overline{A} \Leftrightarrow (A \cap U) \neq \emptyset$ für alle $U \in \mathfrak{U}(f)$. $\boxed{2}$

Aufgabe 4: Sei $(X, \mathcal{T})$ ein topologischer Raum und $\mathfrak{F}$ das System der abgeschlossenen Mengen. Beweisen Sie:

a) Ist $A \subset B \subset X$ mit $B \in \mathfrak{F}$, dann gilt $\overline{A} \subset B$. $\boxed{3}$

b) Für alle $A \subset X$ ist $\overline{A}$ abgeschlossen und $\overline{A}$ ist die kleinste abgeschlossene Teilmenge von X, die A enthält. $\boxed{2}$

c) Ist $A \subset X$ und $f \notin \overline{A}$, dann existiert ein $B \in \mathfrak{F}$ mit $A \subset B$ und $f \notin B$. $\boxed{1}$

d) Für $A \subset X$ gilt
$$\overline{A} = \bigcap_{B \in \mathfrak{F}, B \supset A} B.$$ $\boxed{1}$

$\boxed{\boxed{27}}$

Waclaw Sierpinski[1]

Born: 14 March 1882 in Warsaw, Poland
Died: 21 Oct 1969 in Warsaw, Poland

Waclaw Sierpinski entered the University of Warsaw in 1899. In 1903 the Department of Mathematics and Physics offered a prize for the best essay from a student on Voronoy's contribution to number theory. Sierpinski was awarded the gold medal in the competition.

The results in the prize essay were a major contribution to a famous problem on lattice points. Suppose $R(r)$ denotes the number of points (m, n), $m, n \in \mathbb{Z}$ contained in a circle centre O, radius r. There exists a constant C and a number k with $|R(r) - r^2| < Cr^k$. Let d be the minimal value of k. Gauss proved in 1837 that $d \leq 1$. Sierpinski's major contribution was to show that it was possible to improve the inequality to $d \leq 2/3$. In 1913 Landau shortened Sierpinski's proof and described the result as profound.

Sierpinski graduated in 1904 and worked for a while as a school teacher of mathematics and physics in a girls school in Warsaw. However when the school closed because of a strike, Sierpinski decided to go to Krakv to study for his doctorate. At the Jagiellonian University in Krakv he attended lectures by Zaremba on mathematics, studying in addition astronomy and philosophy. He received his doctorate and was appointed to the University of Lvov in 1908.

In fact it was in 1907 that Sierpinski first became interested in set theory. It happened when he came across a theorem which stated that points in the plane could be specified with a single coordinate. He wrote to Banachiewicz, who was at Gttingen at the time, asking him how such a result was possible. He received a one word reply 'Cantor'. Sierpinski began to study set theory and in 1909 he gave the first ever lecture course devoted entirely to set theory.

When World War I began in 1914, Sierpinski and his family happened to be in Russia. He was interned in Viatka. However Egorov and Luzin heard that he had been interned and arranged for him to be allowed to go to Moscow. Sierpinski spent the rest of the war years in Moscow working with Luzin. Together they began the study of analytic sets. In 1916, during his time in Moscow, Sierpinski gave the first example of an absolutely normal number, that is a number whose digits occur with equal frequency

[1] Aus: 'The MacTutor History of Mathematics archive' der University of St Andrews, Scotland. (gekrzt)

in whichever base it is written. Borel had proved such numbers exit but Sierpinski was the first to give an example.

When World War I ended in 1918, Sierpinski returned to Lvov. However shortly after taking up his appointment again in Lvov he was offered a post at the University of Warsaw which he accepted. In 1919 he was promoted to professor at Warsaw and he spent the rest of his life there.

In 1920 Sierpinski, together with his former student Mazurkiewicz, founded the important mathematics journal Fundamenta Mathematica. From this period Sierpinski worked mostly is in the area of set theory but also on point set topology and functions of a real variable. In set theory he made important contributions to the axiom of choice and to the continuum hypothesis. He studied the Sierpinski curve which describes a closed path which contains every interior point of a square. The length of the curve is infinity, while the area enclosed by it is 5/12 that of the square. Sierpinski continued to collaborate with Luzin on investigations of analytic and projective sets. His work on functions of a real variable include results on functional series, differentiability of functions and Baire's classification.

Sierpinski was also highly involved with the development of mathematics in Poland. He had been honoured with election to the Polish Academy in 1921 and he was made dean of the faculty at the University of Warsaw in the same year. In 1928 he became vice-chairman of the Warsaw Scientific Society and, in the same year was elected chairman of the Polish Mathematical Society.

In 1939 life in Warsaw changed dramatically with the advent of World War II. Sierpinski continued working in the 'Underground Warsaw University' while his official job was a clerk in the council offices in Warsaw. His publications continued since he managed to send papers to Italy. Each of these papers ended with the words:

The proofs of these theorems will appear in the publication of Fundamenta Mathematica

which everyone understood meant 'Poland will survive'.

Sierpinski was the author of the incredible number of 724 papers and 50 books. He retired in 1960 as professor at the University of Warsaw but he continued to give a seminar on the theory of numbers at the Polish Academy of Sciences up to 1967. He received so many honours that it would be impossible to mention them all here.

Rotkiewicz, who was a student of Sierpinski's wrote: *He was a creative mind and liked creative mathematics. He was the greatest and most productive of Polish mathematicians.*