On N-Summations, II.

Dedicated to my friend and collegue Nico Pumplün
on the occasion of his 70th birthday

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ABSTRACT: Given any R-semimodule M equipped with a semitopology $\mathcal{T}$ we construct an N-protosummation $\mathcal{S}^p(\mathcal{T})$ for M. If $\mathcal{T}$ satisfies certain properties then a similar construction leads to an unconditional N-summation $\mathcal{S}(\mathcal{T})$ for M, that is an N-summation for M equipped with the trivial prenorm $M \to \mathbb{D}$ over the N-summation $(\mathbb{D}^N, \sum_{\mathbb{D}})$ for $\mathbb{D}$. Conversely any N-protosummation $\mathcal{S}$ on M gives rise to a topology $\mathcal{T}(\mathcal{S})$. If $\mathcal{S}$ is an unconditional N-summation then $\mathcal{T}(\mathcal{S})$ acquires certain properties.

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0. Introduction

The goal of this paper is to develop a Galois connection between the conglomerate of N-summations (in the sense of [2]) for a given R-semimodule M and the set of semitopologies (in the sense of [2]) on M. It turns out that this requires the replacement of N-summation by the somewhat broader concept of unconditional N-summations.

In §1 we introduce for an R-semimodule equipped with a semitopology $\mathcal{T}$ the concept of unconditional $\mathcal{T}$-summability for elements μ_* of M^N and prove a variety of properties of unconditionally $\mathcal{T}$-summable elements $\mu_* \in M^N$ under certain assumptions on $\mathcal{T}$. Under these conditions on $\mathcal{T}$ the class $S_M^{\mathcal{T}}$ of unconditionally N-summable elements μ_* of M^N together with the map $\sum_M^{\mathcal{T}} : S_M^{\mathcal{T}} \to M$ that assigns to each $\mu_* \in S_M^{\mathcal{T}}$ its $\mathcal{T}$-sum is a weak unconditional $\mathcal{T}$-summation $S(N)$. If $\mathcal{T}$ has the additional property that the addition on M is $\mathcal{T}$-continuous than $S(\mathcal{T})$ turns out to be an unconditional N-summation and thus an N-summation for M if M is given a suitable prenorm whose value cone C is equipped with a suitable N-summation (such a prenorm together with an N-summation for C does always exist).

In §2 we assign to each weak unconditional $\mathcal{T}$-summation $\mathcal{S} = (S_M, \sum_M)$ a closure operator, again denoted by $\mathcal{S}$, and hence a semitopology $\mathcal{T}(\mathcal{S})$. $\mathcal{T}(\mathcal{S})$ has the properties that are required of $\mathcal{T}$ in §1 to make $\mathcal{S}(\mathcal{T})$ a weak unconditional N-summation. The closure operator $A \mapsto \mathcal{S}(A)$ just mentioned is built from the assignment to each subset A of M the subset $A^{\mathcal{S}}$ of M consisting of all elements $\sum_M \mu_*$, where μ_* is in S_M and has the property that for arbitrarily large finite subsets T of N the partial sum $s_T(\mu_*)$ of μ_* over T is in A. $\mathcal{S}(A)$ is then defined as the intersection of all subsets B of M with $A \subseteq B = B^{\mathcal{S}}$.

§3 deals with morphisms $M \to M'$ of R-semimodules with N-protosummations. We show that such a morphism is always continuous with respect to the semitopology (on M and M') defined in §2. The converse is true if this semitopology on M' satisfies a certain separation assumption (UEP).

1 Unconditional N-Summations for Semitopological Semimodules

By a semitopological R-semimodule we mean an R-semimodule equipped with a semitopology $\mathcal{T}$. If the reference to $\mathcal{T}$ needs to be emphasized we speak of a $\mathcal{T}$-semitopological R-semimodule.

Let $P_{fin}(N)$ be equipped with the discrete topology, denote by $P_{fin}^{\omega}(N)$ the Alexandroff compactification of $P_{fin}(N)$ and let $\omega \notin P_{fin}(N)$ and $P_{fin}^{\omega}(N) = P_{fin}(N) \cup \{\omega\}$. Given $\mu_* \in M^N$ let $s_{\Box}(\mu_*)$ be the map $P_{fin}(N) \ni T \mapsto s_T(\mu_*) \in M$. With these notations we extend and replace [2],3.10, to arbitrary elements of M^N (see also [1], p. 262).

Definition 1.1 Let M be a $\mathcal{T}$-semitopological $\mathbb{N}_0$-semimodule. Let furthermore $\mu_* \in M^N$ and suppose that $s^\omega_\square(\mu_*)$ is a continuous extension to $P^\omega_{\text{fin}}(N)$ of $s_\square(\mu_*)$. Then $s^\omega_\omega(\mu_*)$, that is the value of $s^\omega_\square(\mu_*)$ at ω, is called a *$\mathcal{T}$-sum* of μ_* is said to be *$\mathcal{T}$-summable* with $\mathcal{T}$-sum $\sum^{\mathcal{T}}_M(\mu_*)$ if $\sum^{\mathcal{T}}_M(\mu_*)$ is the sole $\mathcal{T}$-sum of μ_*. μ_* is called *unconditionally $\mathcal{T}$-summable* if

(0) for every subclass N' of $N, \mu^{N'}_*$ is $\mathcal{T}$-summable,

(i) for every subclass N' of N and every map $\varphi : N \to N$ the map $\sum^{\mathcal{T}}_M(\mu^{N'\cap\varphi^{-1}}_*)$ given by $N \ni n \mapsto \sum^{\mathcal{T}}_M(\mu^{N'\cap\varphi^{-1}(n)}_*) \in M$ is $\mathcal{T}$-summable and $\sum^{\mathcal{T}}_M(\sum^{\mathcal{T}}_M(\mu^{N'\cap\varphi^{-1}}_*)) = \sum^{\mathcal{T}}_M(\mu^{N'}_*)$. □

The class of unconditionally $\mathcal{T}$-summable elements of M^N is denoted by $S^{\mathcal{T}}_M$ and the map $S^{\mathcal{T}}_M \ni \mu_* \mapsto \sum^{\mathcal{T}}_M(\mu_*) \in M$ is written as $\sum^{\mathcal{T}}_M$. Furthermore the pair $(S^{\mathcal{T}}_M, \sum^{\mathcal{T}}_M)$ is denoted by $\mathcal{S}(\mathcal{T})$. The class of $\mathcal{T}$-summable elements of M^N is written as $S^{p\mathcal{T}}_M$ and the map $S^{p\mathcal{T}}_M \ni \mu_* \mapsto \sum^{\mathcal{T}}_M(\mu_*)$ is denoted by $\sum^{p\mathcal{T}}_M (S^{p\mathcal{T}}_M, \sum^{p\mathcal{T}}_M)$ is denoted by $\mathcal{S}^p\mathcal{T}$). Obviously, $\sum^{\mathcal{T}}_M = \sum^{p\mathcal{T}}_M |S^{\mathcal{T}}_M$.

Lemma 1.2 *Let M be a $\mathcal{T}$-semitopological $\mathbb{N}_0$-semimodule and let N' be any subclass of N. If $\mu_* \in M^N$ is unconditionally $\mathcal{T}$-summable then so is $\mu^{N'}_*$.*

Proof. If N'' is any subclass of N then $(\mu^{N'}_*)^{N''} = \mu^{N'\cap N''}_*$. Hence $\mu^{N'}_*$ satisfies 1.1, (0) and (i). ■

Lemma 1.3 *Let M be a $\mathcal{T}$-semitopological $\mathbb{N}_0$-semimodule and let furthermore $\varphi : N \to N$ be any map. If $\mu_* \in M^N$ is unconditionally T-summable then so is $\sum^{\mathcal{T}}_M(\mu^{\varphi^{-1}}_*)$.*

Proof. By 1.1, (i), If $\sum^{\mathcal{T}}_M(\mu^{\varphi^{-1}}_*)$ is T-summable. Let $N' \subseteq N$. Then $\sum^{\mathcal{T}}_M(\mu^{\varphi^{-1}}_*)^{N'}$ is the map

$$N \ni n \mapsto \begin{cases} \Sigma^{\mathcal{T}}_M(\mu^{\varphi^{-1}(n)}_*) & , \quad n \in N' \\ 0 & , \quad n \notin N'. \end{cases}$$

Thus we have

$$(*) \qquad (\Sigma^{\mathcal{T}}_M(\mu^{\varphi^{-1}}_*))^{N'} = \Sigma^{\mathcal{T}}_M(\mu^{\varphi^{-1}(N)\cap\varphi^{-1}}_*).$$

Due to 1.1, (i), $(\sum_M^{\mathcal{T}}(\mu_*^{\varphi^{-1}}))^{N'}$ is T-summable and hence $\sum_M^{\mathcal{T}}(\mu_*^{\varphi^{-1}})$ satisfies 1.1, (0). Next let $\psi : N \to N$ be any map. The previous argument shows that for every $m \in N$, $(\sum_M^{\mathcal{T}}(\mu_*^{\varphi^{-1}}))^{N'\cap\psi^{-1}(m)}$ is $\mathcal{T}$-summable and that

$$\begin{aligned}\Sigma_M^{\mathcal{T}}((\Sigma_M^{\mathcal{T}}(\mu_*^{\varphi^{-1}}))^{N'\cap\psi^{-1}(m)}) &= \Sigma_M^{\mathcal{T}}(\Sigma_M^{\mathcal{T}}(\mu^{\varphi^{-1}(N'\cap\psi^{-1}(m))\cap\varphi^{-1}}))\\ &= \Sigma_M^{\mathcal{T}}(\mu_*^{\varphi^{-1}(N'\cap\phi^{-1}(m)}) = \Sigma_M^{\mathcal{T}}(\mu_*^{\varphi^{-1}(N')\cap\varphi^{-1}(\psi^{-1}(m))}) =\\ &= \Sigma_M^{\mathcal{T}}(\mu_*^{\varphi^{-1}(N')\cap(\psi\circ\varphi)^{-1}(m)}).\end{aligned}$$

Therefore

$$\Sigma_M^{\mathcal{T}}((\Sigma_M^{\mathcal{T}}(\mu_*^{\varphi^{-1}}))^{N'\cap\psi^{-1}}) = \Sigma_M^{\mathcal{T}}(\mu^{\varphi^{-1}(N')\cap(\psi\circ\varphi)^{-1}}).$$

So another application of $(*)$ leads to

$$\begin{aligned}\Sigma_M^{\mathcal{T}}(\ \Sigma_M^{\mathcal{T}}(((\ \Sigma_M^{\mathcal{T}}(\mu_*^{\varphi^{-1}}))^{N'\cap\psi^{-1}}) &= \Sigma_M^{\mathcal{T}}(\Sigma_M^{\mathcal{T}}(\mu^{\varphi^{-1}(N')\cap(\psi\circ\varphi)^{-1}})) =\\ = \Sigma_M^{\mathcal{T}}(\mu_*^{\varphi^{-1}(N')}) = \Sigma_M^{\mathcal{T}}((\Sigma_M^{\mathcal{T}}(\mu^{\varphi^{-1}(N')\cap\psi^{-1}})) &= \Sigma_M^{\mathcal{T}}((\Sigma_M^{\mathcal{T}}(\mu_*^{\varphi^{-1}}))^{N'}).\end{aligned}$$

Hence $\sum_M^{T}(\mu_*^{\varphi^{-1}})$ satisfies 1.1, (i). ∎

Lemma 1.4 *Let M be a $\mathcal{T}$-semitopological $\mathbb{N}_0$-semimodule. Suppose furthermore that μ_* is unconditionally $\mathcal{T}$-summable and that $\overline{\mu}_* \in$ has the property that there is a bijection χ : supp $\mu_* \to$ supp $\overline{\mu}_*$ with $\mu_n = \overline{\mu}_{\chi(n)}, n \in$ supp μ_*. Then $\overline{\mu}_*$ is unconditionally $\mathcal{T}$-summable and $\sum_M^{\mathcal{T}}(\overline{\mu}_*) = \sum_M^{\mathcal{T}}(\mu_*)$.*

Proof. Put $S :=$ supp μ_*. Then $s_T(\mu_*) = s_{S\cap T}(\mu_*)$ for all $T \in P_{fin}(N)$. Hence for any $T_0 \in P_{fin}(N)$

$$\{s_T(\mu_*) : T_0 \subseteq T \in P_{fin}(N)\} = \{s_{T'}(\mu_*) : S \cap T_0 \subseteq T' \in P_{fin}(N)\}.$$

Let $\overline{T}_0 := \chi(S \cap T_0)$. Then

$$\{s_T(\mu_*) : T_0 \subseteq T \in P_{fin}(N)\} = \{s_{T'}(\overline{\mu}_*) : \overline{T}_0 \subseteq T' \in P_{fin}(N)\}.$$

Hence $\sum_M^{\mathcal{T}}(\mu_*)$ is a $\mathcal{T}$-sum of $\overline{\mu}_*$. The same argument shows that whenever $\overline{m}$ is a $\mathcal{T}$-sum of $\overline{\mu}_*$ then $\overline{m}$ is a $\mathcal{T}$-sum of μ_*. Thus $\overline{\mu}_*$ is $\mathcal{T}$-summable since μ_* is and we have $\sum_M^{\mathcal{T}}(\overline{\mu}_*) = \sum_M^{\mathcal{T}}(\mu_*)$. Obviously $\overline{\mu}_*$ satisfies 1.1, (0), as for any $\overline{N} \subseteq N, \mu_*^{\chi^{-1}(\overline{N}\cap \text{supp } \overline{\mu}_*)}$ and $\overline{\mu}_*^{\overline{N}}$ satisfy the conditions stated for μ_* and $\overline{\mu}_*$ in 1.4. As for 1.1, (i), let $\overline{\varphi} : N \to N$ be any map and let $\overline{N} \subseteq N$. Partition N into the classes

$$\{\chi^{-1}(n) : n \in \overline{N} \cap \text{ supp } \overline{\mu}_* \text{ and } \overline{\varphi}(n) = \overline{\varphi}(\overline{n})\} \qquad , \ \overline{n} \in \overline{\varphi}(\overline{N} \cap \text{supp } \ \overline{\mu}_*),$$

and the complement in N of the union of these classes. This partition is given by some map $\varphi : N \to N$ whose restriction to supp μ_* equals $\overline{\varphi} \circ \chi$. Then both

$$(\triangle) \qquad \overline{\mu}_*^{\overline{\varphi}^{-1}(\overline{n}) \cap \overline{N}} \quad \text{and } \mu_*^{\chi^{-1}(\overline{N} \cap \text{supp } \overline{\mu}_*) \cap \overline{\varphi}^{-1}(\overline{n})} \quad , \overline{n} \in \overline{\varphi}(\overline{N} \cap \text{supp } \overline{\mu}_*),$$

satisfy the conditions stated for μ_* and $\overline{\mu}_*$ in 1.4, while for the remaining elements $\overline{n}$ of N the two maps in $(\triangle)$ equal 0_*. By the above argument we have

$$\Sigma_M^{\mathcal{T}}(\overline{\mu}_*^{\overline{N} \cap \overline{\varphi}^{-1}}) = \Sigma_M^{\mathcal{T}}(\mu_*^{\chi^{-1}(\overline{N} \cap \text{supp } \overline{\mu}_*) \cap \varphi^{-1}})$$

and

$$\Sigma_M^{\mathcal{T}}(\Sigma_M^{\mathcal{T}}(\overline{\mu}_*^{\overline{N} \cap \overline{\varphi}^{-1}})) = \Sigma_M^{\mathcal{T}}\left(\Sigma_M^{\mathcal{T}}(\mu_*^{\chi^{-1}(\overline{N} \cap \text{supp } \overline{\mu}_* \cap \varphi^{-1})})\right) =$$
$$= \Sigma_M^{\mathcal{T}}(\mu_*^{\chi^{-1}(\overline{N} \cap \text{supp } \overline{\mu}_*)}) = \Sigma_M^{\mathcal{T}}(\overline{\mu}_*^N).$$

■

The preceding results were obtained without any conditions imposed on the semitopology $\mathcal{T}$. However, the following statements will require that $\mathcal{T}$ satisfies appropriate conditions.

Proposition 1.5 *Let M be a $\mathcal{T}$-semitopological $\mathbb{N}_0$-semimodule. Then $\mathcal{T}$ is T_1-semitopology if and only if every $\mu_* \in M^{(N)}$ is unconditionally $\mathcal{T}$-summable and $\sum_M^{\mathcal{T}}(\mu_*) = \sum\{\mu_n : n \in \text{supp } \mu_*\}$.*

Proof. Put $T_0 :=$ supp μ_* and $m := \sum\{\mu_n : n \in \text{supp } \mu_*\}$. Then $s_T(\mu_*) = m$ for all $T_0 \subseteq T \in P_{fin}(N)$ and hence m is a $\mathcal{T}$-sum of μ_*. Suppose $\overline{m} \neq m$. If $\mathcal{T}$ is T_1 then there is a neighborhood $\mathcal{N}$ of $\overline{m}$ with $m \notin \mathcal{N}$. Hence $s_T(\mu_*) \notin \mathcal{N}$ for all $T_0 \subseteq T \in P_{fin}(N)$, whence $\overline{m}$ cannot be a $\mathcal{T}$-sum of μ_*. Since for any $N' \subseteq N$, supp $\mu_*^{N'}$ is also finite, 1.1, (0), is satisfied. As for 1.1, (i), let $\varphi : N \to N$ be any map. Then $\mu_*^{N \cap \varphi^{-1}(n)}$ is $\mathcal{T}$-summable for every $n \in N$. Thus $\sum_M^{\mathcal{T}}(\mu_*^{N' \cap \varphi^{-1}})$ exists and has finite support, and is therefore also $\mathcal{T}$-summable. The formula in 1.1, (i), is now a consequence of the associativity of addition in M. Conversely, if $\mathcal{T}$ is not a T_1-semitopology then there are distinct elements m and $\overline{m}$ of M such that every neighborhood $\mathcal{N}$ of $\overline{m}$ contains m. Let $n_0 \in N$ and denote by $\delta_*^{n_0,m} \in M^{(N)}$ the map satisfying $\delta_{n_0}^{n_0,m} = m$ and $\delta_n^{n_0,m} = 0, n \in N \backslash \{n_0\}$. Then $\delta^{n_0,m}$ has both m and $\overline{m}$ as $\mathcal{T}$-sums and hence is not $\mathcal{T}$-summable. ■

The following definition spells out a separation property of the semitopology $\mathcal{T}$ that ensures that the elements of M^N have at most one $\mathcal{T}$-sum. It is obvious that every Hausdorff semitopology has this separation property but it is not clear that the reverse implication is valid.

Definition 1.6 *The semitopoligical $\mathbb{N}_0$-semimodule M is said to have the Unique Extension Property (UEP) of every map $f : P_{fin}(N) \to M$ such that*

(0) $f(\phi) = 0$,

(i) $f(T' \cup T'') = f(T') + f(T'') \qquad T'$ *and* $T'' \in P_{fin}(N)$ *with* $T' \cap T'' = \phi$

has at most one continuous extension for $P^{\omega}_{fin}(N)$.

Note that a map $f; P_{fin}(N) \to M$ satisfies 1.6, (0) and (i), if and only if there is a $\mu_* \in M^N$ with $f(T) = s_T(\mu_*), T \in P_{fin}(N)$.

Lemma 1.7 *Let M be a $\mathcal{T}$-semitopological $\mathbb{N}$-semimodule and suppose that M satisfies (UEP). Then every $\mu_* \in M^{(N)}$ is unconditionally $\mathcal{T}$-summable and $\sum^{\mathcal{T}}_M(\mu_*) = \sum\{\mu_n : n \in \text{supp } \mu_*\}$. In particular, $\mathcal{T}$ is a T_1-semitopology.*

Proof. See proof of 1.5. ■

Lemma 1.8 *Let M be a $\mathcal{T}$-semitopological $\mathbb{N}$-semimodule. Suppose that M satisfies (UEP). Let $\mu'_*, \mu''_* \in S^{\mathcal{T}}_M$ be such that*

$(A'_{\mu'_*,\mu''_*})$ *given any open subset U of M with $\sum^{\mathcal{T}}_M(\mu'_*) + \sum^{\mathcal{T}}_M(\mu''_*) \in U$ there is a $T_0 \in P_{fin}(N)$ with $s_T(\mu'_*) + s_T(\mu''_*) \in U$ for all $T_0 \subseteq T \in P_{fin}(N)$.*

Then $\mu'_* + \mu''_*$ is $\mathcal{T}$-summable and $\sum^{\mathcal{T}}_M(\mu'_* + \mu''_*) = \sum^{\mathcal{T}}_M(\mu'_*) + \sum^{\mathcal{T}}_M(\mu''_*)$. Moreover, if $A(_{\mu'_*,\mu''_*})$ is valid for all $\mu'_*, \mu''_* \in S^{\mathcal{T}}_M$ then $S^{\mathcal{T}}_M$ is closed under addition.

Proof. Since $s_T(\mu'_*, \mu''_*) = s_T(\mu'_*) + s_T(\mu''_*)$ the condition in 1.8 implies that $\sum^{\mathcal{T}}_M(\mu'_*) + \sum^{\mathcal{T}}_M(\mu''_*)$ is a $\mathcal{T}$-sum of $\mu'_* + \mu''_*$. Hence (UEP) shows that $\mu'_* + \mu''_*$ is $\mathcal{T}$-summable. If the second condition is satisfied then 1.2 shows that $(\mu'_* + \mu''_*)^{N'} = \mu'^{N'}_* + \mu''^{N'}_*$ is $\mathcal{T}$-summable for all $N' \subseteq N$ and that

$\sum_M^{\mathcal{T}}(\mu_*'^{N'} + \mu_*''^{N'}) = \sum_M^{\mathcal{T}}(\mu_*'^{N'}) + \sum_M^{\mathcal{T}}(\mu_*''^{N'})$. Next let $\varphi : N \to N$ be any map. Then for any $n \in N$

$$\Sigma_M^{\mathcal{T}}((\mu_*' + \mu_*'')^{N'\cap\varphi^{-1}(n)}) = \Sigma_M^{\mathcal{T}}(\mu_*'^{N'\cap\varphi^{-1}(n)}) + \Sigma_M^{\mathcal{T}}(\mu_*''^{N'\cap\varphi^{-1}(n)})$$

and thus

$$\begin{aligned} \Sigma_M^{\mathcal{T}}((\mu_*' + \mu_*'')^{N'\cap\varphi^{-1}}) &= \Sigma_M^{\mathcal{T}}((\mu_*')^{N'\cap\varphi^{-1}} + (\mu_*'')^{N'\cap\varphi^{-1}}) \\ &= \Sigma_M^{\mathcal{T}}((\mu_*')^{N'\cap\varphi^{-1}}) + \Sigma_M^{\mathcal{T}}((\mu_*'')^{N'\cap\varphi^{-1}}). \end{aligned}$$

Due to 1.2 and 1.3 both $\sum_M^{\mathcal{T}}((\mu_*')^{N'\cap\varphi^{-1}})$ and $\sum_M^{\mathcal{T}}((\mu_*'')^{N'\cap\varphi^{-1}})$ are unconditionally $\mathcal{T}$-summable and therefore by the above argument $\sum_M^{\mathcal{T}}((\mu_*' + \mu_*'')^{N'\cap\varphi^{-1}})$ is $\mathcal{T}$-summable with $\mathcal{T}$-sum

$$\begin{aligned} &\Sigma_M^{\mathcal{T}}(\Sigma_M^{\mathcal{T}}((\mu_*' + \mu_*'')^{N'\cap\varphi^{-1}})) = \Sigma_M^{\mathcal{T}}(\Sigma_M^{\mathcal{T}}((\mu_*')^{N'\cap\varphi^{-1}}) + \Sigma_M^{\mathcal{T}}((\mu_*'')^{N'\cap\varphi^{-1}})) = \\ &= \Sigma_M^{\mathcal{T}}(\Sigma_M^{\mathcal{T}}((\mu_*')^{N'\cap\varphi^{-1}})) + \Sigma_M^{\mathcal{T}}(\Sigma_M^{\mathcal{T}}((\mu_*'')^{N'\cap\varphi^{-1}})) = \\ &= \Sigma_M^{\mathcal{T}}((\mu_*')^{N'}) + \Sigma_M^{\mathcal{T}}((\mu_*'')^{N'}) = \Sigma_M^{\mathcal{T}}(\mu_*'^{N'} + \mu_*''^{N'}) = \Sigma_M^{\mathcal{T}}((\mu_*' + \mu_*'')^{N'}. \end{aligned}$$

■

The second condition in 1.8, which is $(A_{\mu_*',\mu_*''})$ for all $\mu_*', \mu_*'' \in S_M^{\mathcal{T}}$, is denoted by (A').

Lemma 1.9 *Let M be a $\mathcal{T}$-semitopological $\mathbb{N}_0$-semimodule. Suppose that M satisfies (UEP). Suppose that the following condition holds:*

(A'') for any $\overline{\mu}_*, \overline{\overline{\mu}}_* \in S_M^{\mathcal{T}}$ with supp $\overline{\mu}_* \cap$ supp $\overline{\overline{\mu}}_* = \phi, \overline{\mu}_* + \overline{\overline{\mu}}_*$is in $S_M^{\mathcal{T}}$ and $\Sigma_M^{\mathcal{T}}(\overline{\mu}_* + \overline{\overline{\mu}}_*) = \Sigma_M^{\mathcal{T}}(\overline{\mu}_*) + \Sigma_M^{\mathcal{T}}(\overline{\overline{\mu}}_*)$.

Then for any $\mu_*'', \mu_*'' \in S_M^{\mathcal{T}}, \mu_*' + \mu_*''$ is in $S_M^{\mathcal{T}}$ and $\sum_M^{\mathcal{T}}(\mu_*' + \mu_*'') = \sum_M^{\mathcal{T}}(\mu_*'') + \sum_M^{\mathcal{T}}(\mu_*'')$.

Proof. Let $\mu_*', \mu_*'' \in S_M^{\mathcal{T}}$. Choose $N = N'\dot{\cup}N''$ such that there are bijections $\chi' : N \to N'$ and $\chi'' : N \to N''$. Define $\overline{\mu}_*', \overline{\overline{\mu}}_*'' \in M^N$ by

$$\overline{\mu}_m := \begin{cases} \mu_n' & , m = \chi'(n) \\ 0 & , m \in \chi''(N) \end{cases} \quad \text{and } \overline{\overline{\mu}}_m := \begin{cases} 0 & , m \in \chi'(N) \\ \mu_n'' & , m = \chi''(n) \end{cases} \quad , m \in N.$$

By 1.4 both $\overline{\mu}_*$ and $\overline{\overline{\mu}}_*$ are in $S_M^{\mathcal{T}}$ and we have $\sum_M^T(\overline{\mu}_*) = \sum_M^T(\mu_*)$ and $\sum_M^T(\overline{\overline{\mu}}_*) = \sum_M^T(\mu''_*)$. Moreover we have supp $\overline{\mu}_* \cap$ supp $\overline{\overline{\mu}}_* = \phi$. By assumption we get $\overline{\mu}_* + \overline{\overline{\mu}}_* \in S_M^{\mathcal{T}}$ and

$$\Sigma_M^{\mathcal{T}}(\overline{\mu}_* + \overline{\overline{\mu}}_*) = \Sigma_M^{\mathcal{T}}(\overline{\mu}_*) + \Sigma_M^{\mathcal{T}}(\overline{\overline{\mu}}_*) = \Sigma_M^{\mathcal{T}}(\mu'_*) + \Sigma_M^{\mathcal{T}}(\mu''_*).$$

Define $\varphi : N \to N$ by

$$\varphi(m) := \begin{cases} \chi'^{-1}(m) & , m \in \chi'(N) \\ \chi''^{-1}(m) & , m \in \chi''(N) \end{cases} \qquad , m \in N.$$

Then $\varphi^{-1}(n) = \{\chi'(n), \chi''(n)\}$ and hence, for any $n \in N$,

$$(+) \qquad \overline{\mu}_*^{\varphi^{-1}(n)} = \mu'_n \delta_*^n \quad \text{and } \overline{\overline{\mu}}_*^{\varphi^{-1}(n)} = \mu''_n \delta_*^n.$$

Thus we obtain

$$\Sigma_M^{\mathcal{T}}(\overline{\mu}_*^{\varphi^{-1}}) = \mu'_* \text{ and } \Sigma_M^{\mathcal{T}}(\overline{\overline{\mu}}_*^{\varphi^{-1}}) = \mu''_*$$

and therefore

$$\mu'_* + \mu''_* = \Sigma_M^{\mathcal{T}}(\overline{\mu}_*^{\varphi^{-1}}) + \Sigma_M^{\mathcal{T}}(\overline{\overline{\mu}}_*^{\varphi^{-1}}).$$

However, (+) together with 1.5 and 1.7 show that the latter equals $\sum_M^{\mathcal{T}}(\overline{\mu}_*^{\varphi^{-1}} + \overline{\overline{\mu}}_*^{\varphi^{-1}})$. So

$$\mu'_* + \mu''_* = \Sigma_M^{\mathcal{T}}(\overline{\mu}_*^{\varphi^{-1}} + \overline{\overline{\mu}}_*^{\varphi^{-1}} = \Sigma_M^{\mathcal{T}}((\overline{\mu}_* + \overline{\overline{\mu}}_*)^{\varphi^{-1}}).$$

Since $\overline{\mu}_* + \overline{\overline{\mu}}_*$ is in $S_M^{\mathcal{T}}$ by assumption, 1.3 shows that the right side of the last equation is in $S_M^{\mathcal{T}}$. Thus we have $\mu'_* + \mu''_* \in S_M^{\mathcal{T}}$ and

$$\begin{aligned} \Sigma_M^{\mathcal{T}}(\mu'_* + \mu''_*) &= \Sigma_M^{\mathcal{T}}(\Sigma_M^{\mathcal{T}}((\overline{\mu}_* + \overline{\overline{\mu}}_*)^{\varphi^{-1}})) = \Sigma_M^{\mathcal{T}}(\overline{\mu}_* + \overline{\overline{\mu}}_*) = \\ &= \Sigma_M^{\mathcal{T}}(\overline{\mu}_*) + \Sigma_M^{\mathcal{T}}(\overline{\overline{\mu}}_*) = \Sigma_M^{\mathcal{T}}(\mu'_*) + \Sigma_M^{\mathcal{T}}(\mu''_*). \end{aligned}$$

■

Addendum 1.10 *Let M be a semitopological $\mathbb{N}_0$-semimodule with (UEP). Then the conditions (A') and (A'') are equivalent.*

Proof. Either condition is equivalent with: $S_M^{\mathcal{T}}$ is closed under addition and $\sum_M^T$ is an additive map. ■

Lemma 1.11 *Let M be a T-semitopological R- semimodule. Suppose that M satisfies (UEP) and that for some $r \in R$ the following condition holds:*

(S_r) given $\mu_* \in S_M^{\mathcal{T}}$ and any open subset U of M with $r\Sigma_M^{\mathcal{T}}(\mu_*) \in U$ there is a $T_0 \in P_{fin}(N)$ with $rs_T(\mu_*) \in U$ for all $T_0 \subseteq T \in P_{fin}(N)$.

Then $r\mu_*$ is in $S_M^{\mathcal{T}}$ and $\Sigma_M^{\mathcal{T}}(r\mu_*) = r\Sigma_M^{\mathcal{T}}(\mu_*)$ for all $\mu_* \in S_M^{\mathcal{T}}$.

Proof. Since $s_T(r\mu_*) = rs_T(\mu_*)$ the condition in 1.11 implies that $r\sum_M^T(\mu_*)$ is a $\mathcal{T}$-sum of $r\mu_*$. Due to (UEP) we obtain that $r\mu_*$ is $\mathcal{T}$-summable for all $\mu_* \in S_M^{\mathcal{T}}$ with T-sum $\sum_M^T(r\mu_*) = r\sum_M^T(\mu_*)$. Let $N' \subseteq N$. Then $\mu_*^{N'}$ is $\mathcal{T}$-summable due to 1.2. Consequently $(r\mu_*)^{N'} = r(\mu_*^{N'})$ is $\mathcal{T}$-summable due to the first part of this proof and we have

$$\Sigma_M^{\mathcal{T}}((r\mu_*)^{N'}) = \Sigma_M^{\mathcal{T}}(r(\mu_*^{N'})) = r\Sigma_M^{\mathcal{T}}(\mu_*^{N'}).$$

Next let $\varphi : N \to N$ be any map. Then for any $n \in N$

$$\Sigma_M^{\mathcal{T}}((r\mu_*)^{N'\cap\varphi^{-1}(n)}) = r\Sigma_M^{\mathcal{T}}(\mu_*^{N'\cap\varphi^{-1}(n)})$$

and hence

$$\Sigma_M^{\mathcal{T}}((r\mu_*)^{N'\cap\varphi^{-1}}) = r\Sigma_M^{\mathcal{T}}((\mu_*)^{N'\cap\varphi^{-1}}).$$

By assumption $\sum_M^T((\mu_*)^{N'\cap\varphi^{-1}})$ is $\mathcal{T}$-summable and therefore by the above argument $\sum_M^T((r\mu_*)^{N'\cap\varphi^{-1}})$ is $\mathcal{T}$-summable with $\mathcal{T}$-sum

$$\begin{aligned}\Sigma_M^{\mathcal{T}}(\Sigma_M^{\mathcal{T}}((r\mu_*)^{N'\cap\varphi^{-1}})) &= \Sigma_M^{\mathcal{T}}(r\Sigma_M^{\mathcal{T}}(\mu_*^{N'\cap\varphi^{-1}})) = r\Sigma_M^{\mathcal{T}}(\Sigma_M^{\mathcal{T}}(\mu_*^{N'\cap\varphi^{-1}})) = \\ &= r\Sigma_M^{\mathcal{T}}(\mu_*^{N'}) = \Sigma_M^{\mathcal{T}}(r(\mu_*^{N'})) = \Sigma_M^{\mathcal{T}}(r\mu_*)^{N'}.\end{aligned}$$

■

(S') stands for assumption that (Sr) is satisfied for all $r \in R$.

Definition 1.12 (a) An *N-protosummation* for the $\mathbb{N}_0$-semimodule M is a pair $(S_M, \sum_M)$ consisting of a subclass S_M of M^N and a map $\sum_M$ from S_M to M such that

(0) for every $m \in M$ there is a $n_m \in N$ such that the map $\delta_*^{m,n_m} : N \to N$ given by $\delta_{n_m}^{m,n_m} = m$ and $\delta_n^{m,n_m} = 0, n_m \neq n \in N$, is in S_M and satisfies $\sum_M(\delta_*^{m,n_m}) = m$.

(b) An *unconditional* (resp. *unconditional partial*) *N-summation* for the R-semimodule M is a pair $(S_M, \sum_M)$ consisting of a subclass S_M of M^N and a map $\sum_M : S_M \to M$ such that

(i) $\mathrm{M}^{(N)} \subseteq S_M$ and $\mu_* \in M^{(N)}$ implies $\sum_M(\mu_*) = \sum\{\mu_n : n \in \text{supp } \mu_*\}$,

(ii) for every $\mu_* \in S_M$ (resp. $\mu_* \in M^{(N)}$ and every $\nu_* \in S_M, \mu_* + \nu_*$ is in S_M and $\sum_M \mu_* + \nu_*) = \sum_M(\mu_*) + \sum_M(\nu_*)$,

(ii′) for every $r \in R$ and every $\mu_* \in S_M, r\mu_*$ is in S_M and $\sum_M(r\mu_*) = r\sum_M(\mu_*)$,

(iii) for every $\mu_* \in S_M$ and every map $\varphi : N \to N, \mu_*^{\varphi^{-1}(n)}$ is in S_M for all $n \in N$ and the map $\sum_M(\mu_*^{\varphi^{-1}})$ given by $N \ni n \mapsto \sum_M(\mu_*^{\varphi^{-1}(n)}) \in M$ is in S_M and satisfies $\sum_M(\sum_M(\mu_*^{\varphi^{-1}})) = \sum_M(\mu_*)$.

Proposition 1.13 *Let $(S_M, \sum_M)$ be an N-summation for the prenormed R-semimodule M with value cone C and N-summation $(S_C, \sum_C)$ for C. Then $(S_M, \sum_M)$ is an unconditional N-summation for M. Conversely if $(S_M, \sum_M)$ is an unconditional N-summation for the R-semimodule M then $(S_M, \sum_M)$ is an N-summation for M equipped with the trivial prenorm $\|\Box\|_t : M \to \mathbb{D}$ and the N-summation $(\mathbb{D}^N, \sum_{\mathbb{D}})$ for $\mathbb{D}$. Here*

$$\begin{array}{ll} \mathbb{D} = \{0,1\} \text{ with } 0 < 1, a+b = \sup\{a,b\}, ab = \inf\{a,b\}, & \\ \|m\|_t := \begin{cases} 0 & \text{if } m = 0 \\ 1 & \text{if } m \neq 0 \end{cases} & , m \in M, \\ \sum_{\mathbb{D}}(\mu_*) = \sup\{\mu_n : n \in N\} & , \mu_* \in \mathbb{D}^N \end{array}$$

Proof. The first assertion follows by comparing [2], 3.3, with 1.12. The second assertion follows by straight forward calculation. ■

Tying 1.12 together with 1.8 - 1.10 we obtain

Theorem 1.14 *Let M be a $\mathcal{T}$-semitopological R-semimodule that satisfies (UEP). If both (A') and (S') are valid then $\mathcal{S}(\mathcal{T})$ is an unconditional N-summation for M.*

Proposition 1.15 *Let M be a $\mathcal{T}$-semitopological R-semimodule that satisfies(UEP). If for every $\overline{m} \in M$ the map $M \ni m \mapsto m + \overline{m} \in M$ is continuous then for any $\mu_* \in S_M^{\mathcal{T}}$ and any $\overline{\mu}_* \in M^{(N)}, \mu_* + \overline{\mu}_*$ is in $S_M^{\mathcal{T}}$ and $\sum_M^{\mathcal{T}}(\mu_* + \overline{\mu}_*) = \sum_M^{\mathcal{T}}(\mu_*) + \sum_M^{\mathcal{T}}(\overline{\mu}_*)$. In particular, for any $\mu_* \in S_M^{\mathcal{T}}$ and any $T \in P_{fin}(N), \sum_M^{\mathcal{T}}(\mu_*) = s_T(\mu_*) + \sum_M^{\mathcal{T}}(\mu_*^{N \setminus T})$.*

Proof. Let $\overline{m} := \sum_M^T(\overline{\mu}_*)$ and let U be any open subset of M with $\sum_M^T(\mu_*)+\overline{m} \in U$. Then $V := \{m : m+\overline{m} \in U\}$ is an open subset of M satisfying $\sum_M^T(\mu_*) \in V$. Hence there is a $T_0' \in P_{fin}(N)$ such that $s_T(\mu_*) \in V$ for all $T_0' \subseteq T \in P_{fin}(N)$. Put $\overline{\overline{T}}_0 := \text{supp}\ \overline{\mu}_*$. Then $T_B := T_0' \cup \overline{\overline{T}}_0 \in P_{fin}(N)$ and

$$s_T(\mu_* + \overline{\mu}_*) = s_T(\mu_*) + s_T(\overline{\mu}_*) = s_T(\mu_*) + \overline{m} \in U \quad ,T_0 \subseteq T \in P_{fin}(N).$$

Thus $\sum_M^{\mathcal{T}}(\mu_*) + \overline{m}$ is a $\mathcal{T}$-sum for $\mu_* + \overline{\mu}_*$. Since M satisfies (UEP) we conclude that $\mu_* + \overline{\mu}_*$ is $\mathcal{T}$-summable with $\mathcal{T}$-sum $\sum_M^T(\mu_*)+\overline{m} = \sum_M^T(\mu_*)+\sum_M^T(\overline{\mu}_*)$, proving the formula at the end of 1.15. In order to prove 1.1, (i), let $\varphi : N \to N$ be any map. Then

$$\Sigma_M^{\mathcal{T}}((\mu_* + \overline{\mu}_*)^{N'\cap\varphi^{-1}(n)}) = \Sigma_M^{\mathcal{T}}(\mu_*^{N'\cap\varphi^{-1}(n)} + \overline{\mu}_*^{N'\cap\varphi^{-1}(n)})$$

is well defined for every $n \in N$. Since $\sum_M^T(\overline{\mu}_*^{N'\cap\varphi^{-1}})$ has finite support we have

$$\begin{aligned}
&\Sigma_M^{\mathcal{T}}(\Sigma_M^{\mathcal{T}}((\mu_* + \overline{\mu}_*)^{N'\cap\varphi^{-1}}) = \Sigma_M^{\mathcal{T}}(\Sigma_M^{\mathcal{T}}(\mu_*^{N'\cap\varphi^{-1}} + \overline{\mu}_*^{N\cap\varphi^{-1}})) = \\
&\quad = \Sigma_M^{\mathcal{T}}(\Sigma_M^{\mathcal{T}}(\mu_*^{N'\cap\varphi^{-1}}) + \Sigma_M^{\mathcal{T}}(\overline{\mu}_*^{N'\cap\varphi^{-1}})) = \\
&\quad = \Sigma_M^{\mathcal{T}}(\Sigma_M^{\mathcal{T}}(\mu_*^{N'\cap\varphi^{-1}})) + \Sigma_M^{\mathcal{T}}(\Sigma_M^{\mathcal{T}}(\overline{\mu}_*^{N'\cap\varphi^{-1}})) = \Sigma_M^{\mathcal{T}}(\mu_*^{N'}) + \Sigma_M^{\mathcal{T}}(\overline{\mu}_*^{N'}) = \\
&\quad = \Sigma_M^{\mathcal{T}}(\mu_*^{N'} + \overline{\mu}_*^{N'}) = \Sigma_M^{\mathcal{T}}((\mu_* + \overline{\mu}_*)^{N'}).
\end{aligned}$$

■

We end this section with two statements involving the conditions (A') and (S_r).

Proposition 1.16 *Let M be a $\mathcal{T}$-semitopological R-semimodule. Let furthermore $r \in R$. If the map $M \ni m \mapsto rm \in M$ is continuous then M satisfies (S_r).*

Proof. Let $\mu_* \in S_M^{\mathcal{T}}$ and let U be any open subset of M with $r\sum_M^T(\mu_*) \in U$. Put $V := \{m \in M : rm \in U\}$. Since the map in 1.16 in continuous V is an open subset of M and we have $\sum_M^T(\mu) \in V$. Since μ_* is T-summable there is a ${}_0 \in P_{\text{fin}}(N)$ with $S_T(\mu_*) \in V$ for all $T_0 \subseteq T \in P_{\text{fin}}(N)$. Hence $rs_T(\mu_*) \in U$ for all $T_0 \subseteq T \in P_{\text{fin}}(N)$. ■

Proposition 1.17 *Let M be a $\mathcal{T}$-semitopological R-semimodule. Then M satisfies (A'), provided that either one of the following two conditions is valid:*

(i) M^2 *carries a semitopology such that*

(a) $M^2 \ni (m', m'') \mapsto m' + m'' \in M$ *is continuous,*

(b) *for every open subset* U *of* M^2 *and all* $\mu'_*, \mu''_* \in S_M^{\mathcal{T}}$ *with* $(\sum_M^{\mathcal{T}}(\mu'_*), \sum_M^{\mathcal{T}}(\mu''_*)) \in U$ *there is a* $T_0 \in P_{\text{fin}}(N)$ *with* $(s_T(\mu'_{(}), s_T(\mu''_*)) \in U$ *for all* $T_0 \subseteq T \in P_{\text{fin}}(N)$*;*

(ii) *with* M^2 *carrying the product semitopology,* $M^2 \ni (m', m'') \mapsto m' + m'' \in M$ *is continuous.*

Proof. Suppose (i) is valid. Let U be any open subset of M and let $\mu'_*, \mu''_* \in S_M^{\mathcal{T}}$ satisfy $\sum_M^{\mathcal{T}}(\mu'_*) + \sum_M^{\mathcal{T}}(\mu''_*) \in U$. Put $V := \{(m', m'') \in M^2 : m' + m'' \in U\}$. Then V is an open subset of M^2 with $(\sum_M^{\mathcal{T}}(\mu'_*), \sum_M^{\mathcal{T}}(\mu''_*)) \in V$. Hence (i), (b), furnishes a $T_0 \in P_{\text{fin}}(N)$ with $(s_T(\mu'), s_T(\mu'')) \in V$ for all $T_0 \subseteq T \in P_{\text{fin}}(N)$. Thus $s_T(\mu'_*) + s_T(\mu''_*) \in U$ for all $T_0 \subseteq T \in P_{\text{fin}}(N)$ as had to be shown. the proof using (ii) instead of (i) works similarly. ∎

Remark 1.18 Let M be a $\sum_M^{\mathcal{T}}$-semitopological R-semimodule. Then the initial semitopology on M^2 for which $M^2 \ni (m', m'') \mapsto m' + m'' \in M$ is continuous has as its open sets precisely the sets $\{(m', m'') : m + m'' \in V\}$, where V is some open set of M.

2 The Topology Associated with a N-Protosummation

We begin with a construction on M^k, k any positive integer, where M is a $\mathbb{N}$-semimodule with N-protosummation $\mathcal{S} = (S_M, \sum_M)$. Let $A \subseteq M^k$ and put

$$A^{\mathcal{S}} := \{\ m^1, \ldots, m^k) \in M^k : \text{ there are } \mu_*^1, \ldots, \mu_*^k \in S_M \text{ and a cofinal subclass } P \text{ of } P_{\text{fin}}(N) \text{ such that } m^1 = \Sigma_M(\mu_*^1), \ldots, m^k = \Sigma_M(\mu_*^k) \text{ and } (s_T(\mu_*^1), \ldots, s_T(\mu_*^k)) \in A \text{ for all } T \in P\}.$$

Lemma 2.1 *Let* M *be a* $\mathbb{N}_0$*-semimodule with* N*-protosummation* $\mathcal{S}$*. Let furthermore* A *and* B *subsets of* M^k*. Then*

(0) $\psi^{\mathcal{S}} = \phi$ *and* $(M^k)^{\mathcal{S}} = M^k$*,*

(i) $A \subseteq A^{\mathcal{S}}$,

(ii) $A \subsetneqq B$ *implies* $A^{\mathcal{S}} \subseteq B^{\mathcal{S}}$.

Proof. (0) and (ii) are obvious. As for (i), et $(m^1, \ldots, m^k) \in A$ and denote by μ_*^k the element $\delta_*^{m^k, n_{m^k}}$ of S_M (see 1.12, (a)), $k = 1, \ldots, k$. Put $P := \{T \in P_{\text{fin}}(N) : \{n_{m^1}, \ldots, n_{m^k}\} \subseteq T\}$. Then $m^K = \sum_M(\mu_*^K) = s_T(\mu_*^K)$ for all $T \in P$ and $K = 1, \ldots, k$. ■

$A \subseteq M^k$ is called $\mathcal{S}$-closed precisely when $A^{\mathcal{S}}$ holds. The complement in M^k of a $\mathcal{S}$-closed subset of M^k is said to be $\mathcal{S}$-open.

Lemma 2.2 *Let M be an $\mathbb{N}_0$-semimodule with N-protosummation $\mathcal{S}$. Let furthermore $\{A_i : i \in I\}$ be a family of $\mathcal{S}$-closed subsets of M^k. Then $\cap\{A_i : i \in I\}$ is also $\mathcal{S}$-closed. In particular, for any subset A of M^k there is a smallest $\mathcal{S}$-closed subset $\mathcal{S}(A)$ of M^k that contains A.*

Proof. Let $(m^1, \ldots, m^k) \in (\cap\{A_i : i \in I\})^{\mathcal{S}}$. Then there are $\mu_*^1, \ldots, \mu_*^k \in \mathcal{S}_M$ and a cofinal subclass P of $P_{\text{fin}}(N)$ with $m^K = \sum_M(\mu_*^K), K = 1, \ldots, k$ and $(s_T(\mu_*^1), \ldots, s_T(\mu_*^k)) \in \cap\{A_i : i \in I\}$ for all $T \in P$. Hence $(m^1, \ldots, m^k)$ is also in $A_i^{\mathcal{S}} = A_i, i \in I$, and thus in $\cap\{A_i : i \in I\}$. ■

Lemma 2.3 *Let M be an $\mathbb{N}_0$-semimodule with N-protosummation $\mathcal{S}$. Let furthermore $\{A_1, \ldots, A_p$ be a finitely many $\mathcal{S}$-closed subsets of M^k. Then $A_1 \cup \ldots \cup A_p$ is also $\mathcal{S}$-closed.*

Proof. It suffices to let $p = 2$. Let A and B be $\mathcal{S}$-closed subsets of M^k. If $(m^1, \ldots, m^k)$ is in $(A \cup B)^S$ then there are $\mu_*^1, \ldots, \mu_*^k \in \mathcal{S}_M$ and a confinal subclass P of $P_{\text{fin}}(N)$ such that $m^K = \sum_M(\mu_*^K), K = 1, \ldots, k$ and $(s_T(\mu_*^1), \ldots, s_T(\mu_*^k)) \in A \cup B$ for all $T \in P$. Put $P_A := \{T \in P : (s_T(\mu_*^1), \ldots, s_T(\mu_*^k)) \in A\}$ and define P_B similarly. Then $P = P_A \cup P_B$ whence one of P_A and P_B, say P_A, is cofinal in $P_{\text{fin}}(N)$. Thus $(m^1, \ldots, m^k)$ is in $A^{\mathcal{S}} = A$ and hence in $A \cup B$. ■

Proposition 2.4 *Let M be a $\mathbb{N}_0$-semimodule with N-protosummation $\mathcal{S}$. Let furthermore A and B be subsets of M^k. Then*

(0) $\mathcal{S}(\phi) = \phi$ *and* $S(M^k) = M^k$,

(i) $A \subset \mathcal{S}(A)$,

(ii) $A \subseteq B$ *implies* $\mathcal{S}(A) \subseteq \mathcal{S}(B)$,

(iii) $A\mathcal{S}(\mathcal{S}(A)) = \mathcal{S}(A)$,

(iv) $A = \mathcal{S}(A)$ *if an only if* $A = A^{\mathcal{S}}$.

Proof. Clear from Lemma 2.1 and Lemma 2.2. ■

By Proposition 2.4 the map given by $P(M^k) \ni\mapsto \mathcal{S}(A) \in P(M^k)$ is a closure operator. The associated grid $\mathcal{G}(\mathcal{S})$ (see [2], A.15 and A.16) gives rise to the semitopology $\widehat{\mathcal{G}}(\mathcal{S})$ (see [2], A.3), which on account of 2.3 is in fact a topology $\mathcal{T}^k(\mathcal{S})$. Next we develop some properties of $\mathcal{T}^k(\mathcal{S})$. We shall write $\mathcal{T}(\mathcal{S})$ instead of $\mathcal{T}^1(\mathcal{S})$.

Lemma 2.5 *Let M be a $\mathbb{N}_0$-semimodule with N-protosummation $\mathcal{S}$. Then*

(i) *$A \subseteq M^k$ is $\mathcal{S}$-closed if and only if for every $(\mu_*^1, \dots, \mu_*^k) \in S_M^k$ for which there is a cofinal subclass P of $P_{\text{fin}}(N)$ with $(s_T(\mu_*^1, \dots, s_T(\mu_*^k)) \in A$ for all $T \in P$,) $\sum_M(\mu_*^1), \dots, \sum_M(\mu_*^k)) \in A$ holds;*

(ii) *$U \subseteq M^k$ is $\mathcal{S}$-open if and only if for every $(\mu_*^1, \dots, m_*^k) \in S_M^k$ with $(\sum_M(\mu_*^1), \dots, \sum_M(\mu_*^k)) \in U$ there is a $T_0 \in P_{\text{fin}}(N)$ such that $(s_T(\mu_*^1), \dots, s_T(\mu_*^k)) \in U$ for all $T_0 \subsetneqq T \in P_{\text{fin}}(N)$;*

(iii) *for every subset A of $M^k, A \subseteq A^S \subseteq S(A)$.*

Proof. (i) and (ii) are immediate consequences of the definition of S-open resp. S-closed. (iii) follows from Proposition 2.4, (i). ■

Addendum 2.6 *Let M be a $\mathbb{N}$-semimodule with N-protosummation $\mathcal{S}$. If μ_* is in S_M then for every $\mathcal{T}(\mathcal{S})$-open subset U of M with $\sum_M(\mu_*) \in U$ there is a $T_0 \in P_{\text{fin}}(N)$ such that $s_T(\mu_*) \in U$ for all $T_0 \subseteq T \in P_{\text{fin}}(N)$.*

Proof. This is a special case of Lemma 2.5, (iii). ■

Note that Addendum 2.6 states that for any N-protosummation $\mathcal{S}$ the sum $\sum_M(\mu_*)$ of every element $\mu_* \in S_M$ can be obtained by the limit process (for a suitable topology) described at the beginning of section 1.

Lemma 2.7 *Let M be a $\mathbb{N}_0$-semimodule with N-protosummation $\mathcal{S}$. Let furthermore $\overline{m} \in M$. Then the map $M \ni m \mapsto (\overline{m}, m) \in M^2$ is continuous with respect to the topologies $\mathcal{T}(\mathcal{S})$ and $\mathcal{T}^2(\mathcal{S})$.*

Proof. We have to show that the inverse image of any $\mathcal{S}$-closed subset of M^2 under the above map is $\mathcal{S}$-closed. Let $A \subseteq M^2$ satisfy $A = A^{\mathcal{S}}$. The inverse image of A is the set $B = \{m \in M : (\overline{m}, m) \in A\}$. We have (with the obvious omissions)

$$\begin{aligned} B^{\mathcal{S}} &= \{m' \in M : \text{there is a } m'_* \in S_M \text{ and a cofinal subclass } P \text{ of } P_{\text{fin}}(N) \text{ with} \\ &\qquad m' = \Sigma_M(\mu'_*) \text{ and } s_T(\mu'_*) \in B \text{ for all } T \in P\} = \\ &= \{m' \in M : m' = \Sigma_M(\mu'_*) \text{ and } (\overline{m}, s_T(\mu'_*)) \in A \text{ for all } T \in P\}. \end{aligned}$$

Let $\overline{\mu}_* := \delta_*^{\overline{m}, n_{\overline{m}}}$. Put $\overline{P} := \{T \in P : N_{\overline{m}} \in T\}$. Then $\overline{m} = \sum_M(\overline{\mu}_*)$ and $(s_T(\overline{\mu}_*), s_T(\mu'_*)) \in A$ for all $T \in \overline{P}$. Hence $(\overline{m}, m') \in A^{\mathcal{S}} = A$ and therefore $m' \in B$. ■

Lemma 2.8 *Let M be a $\mathbb{N}_0$-semimodule with N-protosummation $\mathcal{S} = (S_M, \sum_M)$ such that S_M is closed under addition and that $\sum_M$ is an additive map. Then the map $M^k \ni (m^1, \ldots, m^k) \mapsto m^1 + \ldots + m^k, k = 1, 2, \ldots,$ is continuous with respect to the topologies $\mathcal{T}^k(\mathcal{S})$ and $\mathcal{T}(\mathcal{S})$.*

Proof. Let $A \subseteq M$ with $A = A^{\mathcal{S}}$. The inverse image of A is the set $B = \{(m^1, \ldots, m^k) \in M^k : m^1 + \ldots + m^k \in A\}$. We have

$$\begin{aligned} B^{\mathcal{S}} &= \{m^1, \ldots, m^k) \in M^k : \text{there is a } (\mu_*^1, \ldots, \mu_*^k) \in S_M^k \text{ and a cofinal} \\ &\qquad \text{subclass } P \text{ of } P_{\text{fin}}(N) \text{ with } m^K = \sigma_M(\mu_*^K), K = 1, \ldots, k \text{ and} \\ &\qquad (s_T(\mu_*^1), \ldots, s_T(\mu_*^k) \in B \text{ for all } T \in P\} = \\ &= \{(m^1, \ldots, m^k) \in M^k : \text{ there is a } (\mu_*^1, \ldots, m_*^k) \in S_M^k \text{ and a cofinal} \\ &\qquad \text{subclass} P \text{ of } P_{\text{fin}}(N) \text{ with } m^K = \Sigma_M(\mu_*^K), K = 1, \ldots, k, \text{ and} \\ &\qquad s_T(\mu_*^1) + \ldots + s_T(\mu_*^k) \in A \text{ for all } T \in P\}. \end{aligned}$$

By assumption $\mu_*^1 + \ldots + \mu_*^k$ is in S_M. Since $s_T(\mu_*^1 + \ldots + \mu_*^k) = s_T(\mu_*^1) + \ldots + s_T(\mu_*^k) \in A$ for all $T \in P$. We obtain $\sum_M(\mu_*^1 + \ldots + \mu_*^k) \in A^{\mathcal{S}} = A$. Since $\sum_M(\mu_*^1 + \ldots + \mu_*^k) = \sum_M(\mu_*^1) + \ldots + \sum_M(\mu_*^k) = m^1 + \ldots + m^k$ by assumption, it follows that $(m^1, \ldots, m^k)$ is in B. ■

Lemma 2.9 *Let M be a R-semimodule with N-protosummation $\mathcal{S} = (S_M, \sum_M)$ such that S_M is closed under left multiplication with any $r \in R$ and that $\sum_M(r\mu_*) = r\sum_M(\mu_*)$ for all $r \in R$ and $\mu_* \in S_M$. Then for every $r \in R$ the map $M \ni m \mapsto rm \in M$ is continuous with respect to the topology $\mathcal{T}(\mathcal{S})$.*

Proof. Let $A \subseteq M$ with $A = A^{\mathcal{S}}$. Then the inverse image of A is the set $B = \{m \in M : rm \in A\}$. Hence we obtain

$$\begin{aligned} B^{\mathcal{S}} &= \{m \in M : \text{there is a } \mu_* \in S_M \text{ and a cofinal subclass } P \text{ of } P_{\text{fin}}(N) \\ &\quad \text{with } m = \Sigma_M(\mu_*) \text{ and } (s_T(\mu_*) \in B \text{ for all } T \in P\} = \\ &= \{(m \in M : \text{ there is a } (\mu_* \in S_M \text{ and a cofinal subclass} P \text{ of } P_{\text{fin}}(N) \\ &\quad \text{with } m = \Sigma_M(\mu_*), \text{ and } rs_T(\mu_*) \in A \text{ for all } T \in P\}. \end{aligned}$$

Since $r\mu_*$ is in S_M and $s_T(r\mu_*) = rs_T(\mu_*)$ is in A for all $T \in P$ we have $\sum_M(r\mu_*) = r\sum_M(\mu_*)$ in A and thus $m = \sum_M(\mu_*)$ in B.

■

Lemma 2.10 *Let M be a $\mathbb{N}_0$-semimodule with N-protosummation $\mathcal{S}$. Then $\mathcal{T}^k(\mathcal{S})$ is a T_1-topology if and only if every $(\mu_*^1, \ldots, \mu_*^k) \in S_M^k$ for which there is a cofinal subclass P of $P_{\text{fin}}(N)$ and a $(m^1, \ldots, m^k) \in M^k$ such that $(s_T(\mu_*^1), \ldots, s_T(\mu_*^k) = (m^1, \ldots, m^k)$ for all $T \in P$ satisfies $(\sum_M(\mu_*^1), \ldots, \sum_M (\mu_*^k)) = (m^1, \ldots, m^k)$. In particular, if $\mathcal{T}(\mathcal{S})$ is a T_1 topology then so is each $\mathcal{T}^k(\mathcal{S}), k = 2, 3, \ldots$.*

Proof. Suppose that $\mathcal{T}^k(\mathcal{S})$ is a T_1-topology and that $(\mu_1^*, \ldots, \mu_*^k) \in S_M^k$ satisfies the hypotheses stated in Lemma 2.9. Put $\overline{m}^K := \sum_M(\mu_*^K), K = 1, \ldots, k$. If $\overline{U}$ is an $\mathcal{S}$-open subset of M^k containing $(\overline{m}^1, \ldots, \overline{m}^k)$ but not containing $(m^1, \ldots, m^k)$ then $(s_T(mu_*^1), \ldots, s_T(\mu_*^k) \in \overline{U}$ and we have a contraction to Lemma 2.5, (ii). In order to prove the converse let $(m^1, \ldots, m^k) \in M^k$. Then $\{(m^1, \ldots, m^k)\}^{\mathcal{S}}$ consists of all $(\overline{m}^1, \ldots, \overline{m}^k)$ for which there is a $(\overline{\mu}_*^1, \ldots, \overline{\mu}_*^k) \in S_M^k$ and a cofinal subclass P of $P_{\text{fin}}(N)$ such that $\overline{m}^K = \sum_M(\overline{\mu}_*^K), K = 1, \ldots, k$, and $(s_T(\overline{\mu}_*^1), \ldots, s_T(\overline{\mu}_*^k) = (m^1, \ldots, m^k)$ for all $T \in P$. Hence we obtain $(\overline{m}_1, \ldots, \overline{m}_k) = (\sum_M(\overline{\mu}_*^1), \ldots, \sum_M(\overline{\mu}_*^k) = (m^1, \ldots, m^k)$, that is $\{(m^1, \ldots, m^k)\}^{\mathcal{S}} = \{(m^1, \ldots, m^k)\}$. This means that $\mathcal{T}^k(\mathcal{S})$ is a T_1-topology. ■

Proposition 2.11 *Let M be a $\mathbb{N}_0$-semimodule with N-protosummation $\mathcal{S}$ such that $\mathcal{T}(\mathcal{S})$ is a T_1-topology. Then every $\mu_* \in S_M$ satisfies:*

> *if $\overline{m} \in M$ is such that for each $\mathcal{S}$-open subset U of M with $\overline{m} \in U$ there is a $T_0 \in P_{\text{fin}}(N)$ with $s_T(\mu_*) \in U$ for all $T_0 \subseteq T \in P_{\text{fin}}(N)$ then $\overline{m} = \sum_M(\mu_*)$.*

Proof. Suppose there were a $\overline{m}$ contradicting the stated property with respect to some $\mu_* \in S_M$. Since $\mathcal{T}(\mathcal{S})$ is a T_1-topology there is an $\mathcal{S}$-open subset U of M with $\overline{m} \in U$ and $\sum_M(\mu_*) \notin U$ such that $s_T(\mu_*) \in U$ for all $T_0 \subseteq T \in P$, where T_0 is chosen suitably. This, however, is in violation of Lemma 2.5, (ii). ■

This section closes with a construction of $\mathcal{S}(A)$, whee A is any subset of M^k and M is a $\mathbb{N}_0$-semimodule with N-protosummation $\mathcal{S} = (S_M, \sum_M)$. Since $A \subseteq S(A)$ we have $A \subseteq A^{\mathcal{S}} \subseteq (S(A))^{\mathcal{S}} = \mathcal{S}(A)$. Put $A^{\mathcal{S}^1} := A^{\mathcal{S}}$ and define by transfinite induction, for any ordinal η,

$$A^{\mathcal{S}^\eta} := \begin{cases} (A^{\mathcal{S}^{\eta'}})^{\mathcal{S}} & \text{, if } \eta \text{ is a successor ordinal with } \eta = \eta' + 1 \\ \cup\{A^{\mathcal{S}^{\eta'}} : \eta' < \eta\} & \text{, otherwise.} \end{cases}$$

With this notation we obtain

Proposition 2.12 *Let M be a $\mathbb{N}_0$-semimodule with N-protosummation $\mathcal{S}$ and let $A \subseteq M^k$. Then there is an ordinal η_0 with* $\operatorname{card}(\eta_0) \leq \operatorname{card}(M^k)$ *such that $\mathcal{S}(A) = A^{\mathcal{S}^{\eta_0}}$.*

Proof. We have $A \subseteq A^{\mathcal{S}^{\eta'}} \subseteq A^{\mathcal{S}^\eta}$ for any ordinals η' and η with $\eta' \leq \eta$. Hence there is an ordinal η_0 with $\operatorname{card}(\eta_0) \leq \operatorname{card}(M^k)$ and $A^{\mathcal{S}^{\eta_0}} = A^{\mathcal{S}^\eta}$ for all $\eta_0 \leq \eta$. We claim that $A^{\mathcal{S}^\eta} \subseteq \mathcal{S}(A)$ holds for any ordinal η. This is true for $\eta = 1$ due to Lemma 2.5, (iii). If η is a successor ordinal with $\eta = \eta' + 1$ and $A^{\mathcal{S}^{\eta'}} \subseteq \mathcal{S}(A)$ then $A^{\mathcal{S}^\eta} = (A^{\mathcal{S}^{\eta'}})^{\mathcal{S}} \subseteq (\mathcal{S}(A))^{\mathcal{S}} = \mathcal{S}(A)$. If η is not a successor ordinal and $A^{\mathcal{S}^{\eta'}} \subseteq \mathcal{S}(A)$ for all $\eta' < \eta$ then $A^{\mathcal{S}^\eta} = \cup\{A^{\mathcal{S}^{\eta'}} : \eta' < \eta\} \subseteq \mathcal{S}(A)$. This means that we have $A^{\mathcal{S}^{\eta_0}} \subseteq \mathcal{S}(A)$. Hence $A \subseteqq A^{\mathcal{S}^{\eta_0+1}} = (A^{\mathcal{S}^{\eta_0}})^{\mathcal{S}} = A^{\mathcal{S}^{\eta_0}}$ and therefore $\mathcal{S}(A) \subseteq A^{\mathcal{S}^{\eta_0}}$, that is $\mathcal{S}(A) = A^{\mathcal{S}^{\eta_0}}$. ■

3 Morphisms of R-Semimodules with N-Protosummations

Given any map $f : M \to M'$ and any $\mu_* \in M^N$ we denote by $f^N(\mu_*)$ the map $N \ni n \mapsto f(\mu_n) \in M'$. Hence $f^N(\mu_*)$ is in $M^{\mathbb{N}}$.

Definition 3.1 Let M and M' be R-semimodules with N-protosummations $S = (S_M, \sum_M)$ resp. $S' = (S_{M'}, \sum_M)$. Then the homomorphism $f : M \to M'$ of R-semimodules is called a *morphism of R-semimodules with N-protosummation* if

(i) $f^n(S_M) \subseteq S_{M'}$,

(ii) $f(\sum_M(\mu_*)) = \sum_{M'}(f^N(\mu_*)), \qquad \mu_* \in S_M$.

Lemma 3.2 *Let M and M' be $\mathbb{R}$-semimodules with N-protosummation $\mathcal{S} = (S_M, \sum_M)$ resp. $\mathcal{S}' = (S_{M'}, \sum_{M'})$. Let furthermore $f : M \to M'$ be a morphism of R-semimodules with N-protosummations. then* supp $f(\mu_*) \subseteq$ supp μ_* *for all $\mu_* \in S_M$. In particular, if $0_* \in S_M$ then $f(0_*) = 0_* \in S_{M'}$. If S_M is closed under addition and $\sum_M$ is additive then $\sum_{M'}$ is additive on $f^N(S_M)$. If S_M is closed under left multiplication by $r \in R$ and $\sum_M$ commutes with left multiplication by r then so does $\sum_M$, on $f^N(S_M)$.*

Proof. We only check the second assertion. If S_M is closed under addition and $\mu_*, \mu'_* \in S_M$ then $f^N(\mu_*) + f^N(\mu'_*) = f^N(\mu_* + \mu'_*) \in S_{m'}$ and

$$\Sigma_{M'}(f^N(\mu_*) + f^N(\mu'_*)) = \Sigma_{M/}(f^N(\mu_* + \mu'_*)) = f(\Sigma_M(\mu_* + \mu'_*))$$

$$= f(\Sigma_M(\mu_*) + \Sigma_M(\mu'_*)) = f(\Sigma_M(\mu_*)) + f(\Sigma_M(\mu'_*)) = \Sigma_{M'}(f^N(\mu_*)) + \Sigma_{M'}(f^N(\mu'_*)).$$

■

Proposition 3.3 *Let M and M' be $\mathbb{R}$-semimodules with N-protosummation S resp. S'. Let furthermore $f : M \to M'$ be a morphism of R-semimodules with N-protosummations. Then for every $k \in \mathbb{N}$, $f^k : M^k \to M'^k$ is continuous with respect to the topology $\mathcal{T}^k(\mathcal{S})$ resp. $\mathcal{T}^k(\mathcal{S}')$.*

Proof. Let $A \subseteq M^k$ and let $(m^1, \ldots, m^k) \in A^S$. Then there are $\mu^1_*, \ldots, \mu^k_* \in S_M$ and a cofinal subclass P of $P_{\text{fin}}(N)$ with $m^K = \sum_M(\mu^K_*)$, $K = 1, \ldots, k$ and $(s_T(\mu^1_*), \ldots, s_T(\mu^k_*)) \in A$ for all $T \in P$. Hence

$$f(m^K) = f(\Sigma_M(\mu^K_*)) = \Sigma_{M'}(f^N(\mu^K_*)) \qquad , K = 1, \ldots, k,$$

and

$$(s_T(f^N(\mu^1_*), \ldots, s_T(f^N(\mu^k_*)) = f^k(s_T(\mu^1_*), \ldots, s_T(\mu^k_*)) \in f^k(A) \qquad , T \in P.$$

Thus $f(m^1, \ldots, m^k) \in (f^k(A))^{\mathcal{S}'}$ and therefore $f^k(A^{\mathcal{S}}) \subseteq (f^k(A))^{\mathcal{S}'}$.

Now suppose that $A' \subseteq M'^k$ is $\mathcal{T}^k(\mathcal{S}')$-closed. Due to Proposition 2.4, (iv), this means that $A' = A'^{\mathcal{S}'}$. Put $A := (f^k)^{-1}(A')$. If $(m^1, \ldots, m^k) \in A^{\mathcal{S}}$ then

$$(f(m^1), \ldots, f(m^k)) = f^k(m^1, \ldots, m^k) \in A'^{\mathcal{S}'} = A'$$

and therefore $(m^1, \ldots, m^k) \in A$. Thus $A = A^{\mathcal{S}}$, that is A is $\mathcal{T}^k(\mathcal{S})$-closed. ■

Proposition 3.4 *Let M and M' be semitopological R-semimodules with semi-topolgoy $\mathcal{T}$ resp $\mathcal{T}'$ and suppose that M' satisfies (UEP). Let furthermore $f : M \to M'$ be a continuous homomorphism of R-semimodules. Then f is a morphism of R-semimodules with N-protosummations $\mathcal{S}^p(\mathcal{T})$ resp. $\mathcal{S}^p(\mathcal{T}')$ as well as a morphism of R-semimodules with unconditional partial N-summation $\mathcal{S}(\mathcal{T})$ resp. $\mathcal{S}(\mathcal{T}')$.*

Proof. Let $\mu^* \in S_M$ with $m := \sum_M(\mu_*)$. Then $f(s_T(\mu_*)) = s_T(f^N(\mu_*)), T \in P_{\text{fin}}(N)$, whence $f(m)$ si a $\mathcal{T}'$-sum of $f^N(\mu_*)$. Since M' satisfies (UEP), $f^N(\mu_*)$ is $\mathcal{T}$-summable. In particular, Definition 3.1, (i) and (ii), are satisfied. The second part of Proposition 3.4 follows from the formula $f(\sum_M(\mu_*^{N'})) = \sum_{M'}((f^N(\mu_*))^{N'})$. ■

References

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