The Categories of Pro-Lie Groups and Pro-Lie Algebras

For DIETER PUMPLÜN on his 70th birthday

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Abstract. This text records some basic results from a projected monograph on "Lie Theory and the Structure of pro-Lie groups and Locally Compact Groups" which may be considered a sequel to our book "The Structure of Compact Groups" [De Gruyter, Berlin, 1998]. In focus are the categories of projective limits of finite dimensional Lie groups and of projective limits of finite dimensional Lie algebras and their functorial relationship.

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Introduction

There are two prime reasons for the success of the structure and representation theory of locally compact groups: The existence of Haar integral on a locally compact group G and the successful resolution of Hilbert's Fifth Problem with the proof that connected locally compact groups can be approximated by finite dimensional Lie groups. Lie groups themselves have a highly developed structure and representation theory.

Haar measure is the key to the representation theory of compact and locally compact groups on Hilbert space, and the wide field of harmonic analysis with ever so many ramifications (including e.g. abstract probability theory on locally compact groups). A theorem of A. Weil's shows that, conversely, a complete topological group with a left- (or right-) invariant measure is locally compact. Thus the category of locally compact groups is that which is exactly suited for real analysis resting on the existence of an invariant integral. One cannot expect to extend that aspect of locally compact groups to larger classes.

However, from a category theoretical and from a Lie theoretical point of view the class of a locally compact groups has defects which go rather deep. Indeed while every locally compact group G has a Lie algebra $\mathfrak{L}(G)$ and an exponential function $\exp: \mathfrak{L}(G) \to G$, the additive group of the Lie algebra is never locally compact unless it is finite dimensional. Apart from individual studies such as [2,5,7,8], the Lie theory of locally compact groups has never been *systematically* considered or exploited, although a good start was made in [4] for the purpose of a structure theory of *compact* groups.

Thus from the view point of Lie theory, the category of locally compact groups appears to have two major drawbacks:
—The topological abelian group underlying the Lie algebra $\mathfrak{L}(G)$ fails to be locally compact unless $\mathfrak{L}(G)$ is finite dimensional. In other words, the very Lie theory

making the structure theory of locally compact groups interesting leads us outside the class.
—The category of locally compact groups is not closed under the forming of products, even of copies of $\mathbb{R}$; it is not closed under projective limits of projective systems of finite dimensional Lie groups, let alone under arbitrary limits. In other words, the category of locally compact groups is badly incomplete.

Let us denote the category of all (Hausdorff) topological groups and continuous group homomorphisms by $\mathbb{TOPGR}$. It turnes out that the full subcategory pro$\mathbb{LIEGR}$ of $\mathbb{TOPGR}$ consisting of *all* projective limits of finite dimensional Lie groups avoids both of these difficulties. This would perhaps not yet be a sufficient reason for advocating this category if it were not for two facts:

—Firstly, while not every locally compact group is a projective limit of Lie groups, every locally compact group has an open subgroup which is a projective limit of Lie groups, so that, in particular, every connected locally compact group is a projective limit of Lie groups.

—Secondly, the category pro$\mathbb{LIEGR}$ is astonishingly well-behaved. Not only is it a complete category, it is closed under passing to closed subgroups and to those quotients which are complete, and it has a demonstrably good Lie theory.

It is therefore indeed surprising that this class of groups has been little investigated in a systematic fashion.

A serious beginning of such an investigation is made in [6] where it is submitted that a general structure theory of locally compact groups should be based on a good understanding of the category pro$\mathbb{LIEGR}$.

This article presents a crisp overview of some of the central results whose proofs will be detailed and whose background and applications will be discussed in [6].

1. Core results on pro-Lie groups

For a description of some basic results on the theory of projective limits of Lie groups some technical background information appears inevitable.

Definition 1.1. A *projective system* D of topological groups is a family of of topological groups $(C_j)_{j\in J}$ indexed by a directed set J and a family of morphisms $\{f_{jk}\colon C_k \to C_j \mid (j,k) \in J \times J,\ j \le k\}$, such that f_{jj} is always the identity morphism and $i \le j \le k$ in J implies $f_{ik} = f_{ij} \circ f_{jk}$. Then the *projective limit of the system* $\lim_{j\in J} C_j$ is the subgroup of $\prod_{j\in J} C_j$ consisting of all J-tuples $(x_j)_{j\in J}$ for which the equation $x_j = f_{jk}(x_k)$ holds for all $j,k \in J$ such that $j \le k$. □

Every cartesian product of topological groups may be considered as a projective limit. Indeed, if $(G_\alpha)_{\alpha\in A}$ is an arbitrary family of topological groups indexed by an infinite set A, one obtains a projective system by considering J to be the set of finite subsets of A directed by inclusion, by setting $C_j = \prod_{a\in j} G_a$ for $j \in J$, and by letting $f_{jk}\colon C_j \to C_k$ for $j \le k$ in J be the projection onto the partial product. The projective limit of this system is isomorphic to $\prod_{a\in A} G_a$.

There are a few sample facts one should recall about the basis properties of projective limits (see e.g. [1], [2], [9], or [6] 1.27 and 1.33):

Let $G = \lim_{j\in J} G_j$ be a projective limit of a projective system

$$\mathcal{P} = \{f_{jk}: G_k \to G_j \mid (j,k) \in J \times J,\ j \le k\}$$

of topological groups with limit morphisms $f_j: G \to G_j$, and let $\mathcal{U}_j$ denote the filter of identity neighborhoods of G_j, $\mathcal{U}$ the filter of identity neighborhoods of G, and $\mathcal{N}$ the set $\{\ker f_j \mid j \in J\}$. Then $\mathcal{U}$ has a basis of identity neighborhoods $\{f_k^{-1}(U) \mid k \in J, U \in \mathcal{U}_k\}$ and $\mathcal{N}$ is a filter basis of closed normal subgroups converging to 1. If all bonding maps $f_{jk}: G_j \to G_k$ are quotient morphisms and all limit maps f_j are surjective, then the limit maps $f_j: G \to G_j$ are quotient morphisms. The limit G is complete if all G_j are complete.

Definition 1.2. For a topological group G let $\mathcal{N}(G)$ denote the set of closed normal subgroups N such that all quotient groups G/N are finite dimensional real Lie groups. Then $G \in \mathcal{N}(G)$ and G is said to be *a proto-Lie group* if

(1) $\mathcal{N}(G)$ is a filterbasis.

(2) $\mathcal{N}(G)$ converges to 1.

If, furthermore, the following condition (3) is satisfied it is called a *pro-Lie group*:

(3) G is a complete topological group, that is, every Cauchy filter converges. □

The full subcategory of the category $\mathbb{TOPGR}$ of topological groups and continuous homomorphisms consisting of all pro-Lie groups and continuous homomorphisms between them is called pro$\mathbb{LIEGR}$.

Every product of a family of finite dimensional Lie groups $\prod_{j\in J} G_j$ is a pro-Lie group. In particular, $\mathbb{R}^J$ is a pro-Lie group for any set J which is locally compact if and only if the set J is finite. The subgroup

$$\big\{(g_j)_{j\in J} \in \prod_{j\in J} G_j : \{j \in J : g_j \neq 1\} \text{ is finite}\big\}$$

is a proto-Lie group which is not a pro-Lie group if J is infinite and the G_j are nonsingleton. Every proto-Lie group has a completion which is a pro-Lie group. A topological group G is called *almost connected* if the factor group G/G_0 modulo the connected component G_0 of the identity is compact. In the middle of the last century it was proved that every almost connected locally compact group is a pro-Lie group.

Every pro-Lie group G gives rise to a projective system

$$\{p_{NM}: G/M \to G/N : M \supseteq N \text{ in } \mathcal{N}(G)\}$$

whose projective limit it is (up to isomorphism). The converse is a difficult issue, but it is true.

Theorem 1.3. *Every projective limit of Lie groups is a pro-Lie group. Every closed subgroup of a Lie group is a pro-Lie group. Every quotient group of a pro-Lie group is a proto-Lie group and has a completion which is a pro-Lie group.*

Proof. [6], 3.34, 3.35, 4.1; [7]. □

In a topological Lie algebra $\mathfrak{g}$ the filterbasis of closed ideals $\mathfrak{j}$ such that $\dim \mathfrak{g}/\mathfrak{j} < \infty$ i denoted by $\mathcal{I}(\mathfrak{g})$.

Definition 1.4. A topological Lie algebra $\mathfrak{g}$ is called a *pro-Lie algebra* (short for *profinite dimensional Lie algebra*) if $\mathcal{I}(\mathfrak{g})$ converges to 0 and if $\mathfrak{g}$ is a complete topological vector space. □

Under these circumstances, $\mathfrak{g} \cong \lim_{\mathfrak{j}\in\mathcal{I}(\mathfrak{g})} \mathfrak{g}/\mathfrak{j}$, and the underlying vector space is a weakly complete topological vector space, that is is the dual of a real vector space with the weak star topology. For a systematic treatment of the duality of vector spaces and weakly complete topological vector spaces we refer to [4], pp. 319ff. The category of pro-Lie algebras and continuous vector space morphisms is denoted pro$\mathbb{LIEALG}$.

Theorem 1.5. (i) *The category* pro$\mathbb{LIEGR}$ *of pro-Lie groups is closed in* $\mathbb{TOPGR}$ *under the formation of all limits and is therefore complete. It is the smallest full subcategory of* $\mathbb{TOPGR}$ *that contains all finite dimensional Lie groups and is closed under the formation of all limits.*

(ii) *The category* pro$\mathbb{LIEALG}$ *of pro-Lie algebras is closed in the category of topological Lie algebras under the formation of all limits and is therefore complete. It is the smallest category that contains all finite dimensional Lie algebras and is closed under the formation of all limits.*

Proof. [6], 3.3, 3.36; [7]. □

Definition 1.6. A topological group G is said to *have a Lie algebra* $\mathfrak{L}(G)$ if the space $\mathrm{Hom}(\mathbb{R}, G)$ of all continuous one parameter subgroups (i.e. morphisms of topological groups) $X\colon \mathbb{R} \to G$ has a continuous addition and bracket multiplication making it into a topological Lie algebra in such a fashion that

$$(X+Y)(r) = \lim_{n\to\infty} X(\frac{r}{n})Y(\frac{r}{n})$$

and

$$[X,Y](r^2) = \lim_{n\to\infty} X(\frac{r}{n})Y(\frac{r}{n})X(\frac{r}{n})^{-1}Y(\frac{r}{n})^{-1}.$$

If G has a Lie algebra, set $\exp X = X(1)$ and $\exp(r{\cdot}X) = X(r)$ and call

$$\exp\colon \mathfrak{L}(G) \to G$$

the exponential function of G.

Theorem 1.7. *Every pro-Lie group G has a pro-Lie algebra as Lie-algebra, and the assignment $\mathfrak{L}$ which associates with a a pro-Lie group G its pro-Lie algebra is a limit preserving functor.*

Proof. Chapters 2 and 3. □

In fact, a portion of this set-up allows a considerable improvement which we summarize in the next section.

2. The Category Theoretical Version of Lie's Third Theorem

Definition 2.1. A pro-Lie group is said to be *prosimply connected* if $\mathcal{N}(G)$ contains a cofinal subset $\mathcal{NS}(G)$ such that G/N is simply connected for every $N \in \mathcal{NS}(G)$. □

This turns out to be the right concept of simple connectivity for pro-Lie groups in all respects, and it reduces correctly to simple connectivity in the case of finite dimensional Lie groups.

Theorem 2.2. (Lie's Third Theorem for pro-Lie groups) *The Lie algebra functor $\mathfrak{L}\colon \mathrm{pro}\mathbb{LIEGR} \to \mathrm{pro}\mathbb{LIEALG}$ has a left adjoint Γ. It associates with every pro-Lie algebra $\mathfrak{g}$ a unique prosimply connected pro-Lie group $\Gamma(\mathfrak{g})$ and a natural isomorphism $\eta_{\mathfrak{g}}\colon \mathfrak{g} \to \mathfrak{L}\big(\Gamma(\mathfrak{g})\big)$ such that for every morphism $\varphi\colon \mathfrak{g} \to \mathfrak{L}(G)$ there is a unique morphism $\varphi'\colon \Gamma(\mathfrak{g}) \to G$ such that $\varphi = \mathfrak{L}(\varphi') \circ \eta_{\mathfrak{g}}$.*

Proof. [6], 5.5 and 5.6. □

In fact, for each pro-Lie algebra $\mathfrak{g}$ $\Gamma(\mathfrak{g})$ is a projective limit of a projective system of simply connected Lie groups. The fact that $\mathfrak{L}$ is a right adjoint confirms its propty of preserving all limits.

Remarkably, $\mathfrak{L}$ preserves some colimits as well:

Theorem 2.3. *The functor $\mathfrak{L}$ preserves quotients. Specifically, assume that G is a pro-Lie group and N a closed normal subgroup and denote by $q\colon G \to G/N$ the quotient morphism. Then G/N is a proto-Lie group whose Lie algebra $\mathfrak{L}(G/N)$ is a pro-Lie algebra and the induced morphism of pro-Lie algebras $\mathfrak{L}(q)\colon \mathfrak{L}(G) \to \mathfrak{L}(G/N)$ is a quotient morphism. The exact sequence*

$$0 \to \mathfrak{L}(N) \to \mathfrak{L}(G) \to \mathfrak{L}(G/N) \to 0$$

induces an isomorphism $X + \mathfrak{L}(N) \mapsto \mathfrak{L}(f)(X) : \mathfrak{L}(G)/\mathfrak{L}(N) \to \mathfrak{L}(G/N)$. □

The core of Theorem 2.3 is proved by showing that for every quotient morphism $f\colon G \to H$ of topological groups, where G is a pro-Lie group, every one parameter subgroup $Y\colon \mathbb{R} \to H$ lifts to one of G, that is, there is a one parameter subgroup σ of G such that $Y = f \circ \sigma$. ([6], 4.19, 4.20.) This requires the Axiom of Choice.

Corollary 2.4. *Let G be a pro-Lie group. Then $\{\mathfrak{L}(N) \mid N \in \mathcal{N}(G)\}$ converges to zero and is cofinal in the filter $\mathcal{I}(\mathfrak{L}(G))$ of all ideals $\mathfrak{i}$ such that $\mathfrak{L}(G)/\mathfrak{i}$ is finite dimensional.*

Furthermore, $\mathfrak{L}(G)$ is the projective limit $\lim_{N\in\mathcal{N}(G)} \mathfrak{L}(G)/\mathfrak{L}(N)$ of a projective system of bonding morphisms and limit maps all of which are quotient morphisms, and there is a commutative diagram

$$\begin{array}{ccccc}
\mathfrak{L}(G) & \xrightarrow{\mathfrak{L}(\gamma_G)} & \mathfrak{L}(\lim_{N\in\mathcal{N}(G)} \frac{G}{N}) & \cong & \lim_{n\in\mathcal{N}(G)} \frac{\mathfrak{L}(G)}{\mathfrak{L}(N)} \\
{\scriptstyle \exp_G}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathfrak{L}(\lim_{N\in\mathcal{N}(G)} \exp_{G/N})} & & \\
G & \xrightarrow[\gamma_G]{} & \lim_{N\in\mathcal{N}(G)} G/N. & &
\end{array}$$

Proof. [6], 4.21. □

Theorem 2.3 expresses a version of exactness of $\mathfrak{L}$. But there is also an exactness theorem for Γ.

Theorem 2.5. *If $\mathfrak{h}$ is a closed ideal of a pro-Lie algebra $\mathfrak{g}$, then the exact sequence*

$$0 \to \mathfrak{h} \xrightarrow{\ i\ } \mathfrak{g} \xrightarrow{\ q\ } \mathfrak{g}/\mathfrak{h} \to 0$$

induces an exact sequence

$$1 \to \Gamma(\mathfrak{h}) \xrightarrow{\Gamma(j)} \Gamma(\mathfrak{g}) \xrightarrow{\Gamma(q)} \Gamma(\mathfrak{g}/\mathfrak{h}) \to 1,$$

in which $\Gamma(j)$ is an algebraic and topological embedding and $\Gamma(q)$ is a quotient morphism.

Proof. [6], 5.7, 5.8, and 5.9. □

3. Quotient preservation by the Lie algebra functor

There are some notworthy consequences of Theorem 2.3.

Proposition 3.1. *Any quotient morphism $f\colon G \to H$ of pro-Lie groups onto a finite dimensional Lie group is a locally trivial fibration.*

Proof. [6], 4.22 (iv). □

For a topological group let $\mathbb{E}(G)$ denote the subgroup generated by all one-parameter subgroups, that is

$$\mathbb{E}(G) \stackrel{\text{def}}{=} \langle \exp \mathfrak{L}(G) \rangle.$$

Proposition 3.2. (i) *For a pro-Lie group G, the subgroup $\mathbb{E}(G)$ is dense in G_0, i.e. $\overline{\mathbb{E}(G)} = G_0$. In particular, a connected nonsingleton pro-Lie group has nontrivial one parameter subgroups.*

Proof. [6], 4.22(i). □

Corollary 3.3. *For a pro-Lie group G the following statements are equivalent:*
(a) *G is totally disconnected.*
(b) $\mathfrak{L}(G) = \{0\}$.
(c) *The filter basis of open normal subgroups of G converges to* 1.

Proof. [6], 4.22. □

We note that for any pro-Lie group G, the additive group of its Lie algebra $\mathfrak{L}(G)$ is also a pro-Lie group. So for an abelian pro-Lie group G, the exponential function $\exp\colon \mathfrak{L}(G) \to G$ is in fact a morphism of pro-Lie groups, and the underlying additive group of $\mathfrak{L}(G)$ is the group $\Gamma\big(\mathfrak{L}(G)\big)$. All of this applies, in particular, to locally compact abelian groups and, in particular, to compact abelian groups. In [4], Chapters 7 and 8, one finds the information that for a compact abelian group G, the kernel of the exponential function $\exp\colon \mathfrak{L}(G) \to G$ is naturally isomorphic to the fundamental group $\pi_1(G)$, and that the image of exp is the arc component G_a of 1 in G. Thus there is a bijective morphism $\mathfrak{L}(G)/\pi_1(G) \to G_a$. It is proved in [8] and in [6], 4.10ff. that for a compact connected abelian group G this morphism is an isomorphism iff in the character group $\widehat{G}$ of G every finite rank pure subgroup is a free direct summand. Whenever this condition is satisfied, G_a is a quotient of the pro-Lie group $\mathfrak{L}(G)$ and this quotient is incomplete if G is not arcwise connected. The simplest such example is the character group G of the discrete group $\mathbb{Z}^{\mathbb{N}}$. In this case $\mathfrak{L}(G) \cong \mathrm{Hom}(\mathbb{Z}^{\mathbb{N}}, \mathbb{R}) \cong \mathbb{R}^{2^{\aleph_0}}$, and this vector group is a simple example of a pro-Lie group with an incomplete quotient group.

Quotients of pro-Lie groups, after all of this, are a somewhat delicate matter. It is therefore good to have sufficient conditions for quotients to be complete, such as for instance in the following theorem.

Theorem 3.4. *The quotient of an almost connected pro-Lie group modulo an almost connected closed normal subgroup is a pro-Lie group.*

Proof. [6], 4.28. □

4. Core results on pro-Lie algebras

In view of the functorial correspondence set up between the categories $\mathrm{pro}\mathbb{LIEGR}$ and $\mathrm{pro}\mathbb{LIEALG}$ every piece of information on pro-Lie algebras translates at once into information on pro-Lie groups; this translation process is often referred to as *Lie Theory.* Chapter 7 of [6] gives details on the workings of a Lie theory of pro-Lie groups. It is this Lie theory of pro-Lie groups that calls for a thorough understanding of the fine structure of pro-Lie algebras in the first place.

Definition 4.1. A pro-Lie algebra $\mathfrak{g}$ is called *semisimple* if it is isomorphic to a product $\prod_{j\in J}\mathfrak{s}_j$ of a family of finite dimensional simple real Lie algebras $\mathfrak{s}_j$. A pro-Lie algebra $\mathfrak{g}$ is called *reductive* iff it is isomorphic to a product of an abelian algebra $\mathbb{R}^I$ for a set I and a semisimple algebra $\mathfrak{s}$. □

Definition 4.2. For subsets $\mathfrak{a}$ and $\mathfrak{b}$ of a Lie algebra $\mathfrak{g}$ let $[\mathfrak{a},\mathfrak{b}]$ denote the linear span of all commutator brackets $[X,Y]$ with $X\in\mathfrak{a}$ and $Y\in\mathfrak{b}$. Inductively, define $\mathfrak{g}^{(1)}=\mathfrak{g}^{[1]}=[\mathfrak{g},\mathfrak{g}]$ and $\mathfrak{g}^{(n+1)}=[\mathfrak{g}^{(n)},\mathfrak{g}^{(n)}]$, $\mathfrak{g}^{[n+1]}=[\mathfrak{g},g^{[n]}]$. A Lie algebra $\mathfrak{g}$ is said to be *countably solvable* if $\bigcap_{n=1}^{\infty}\mathfrak{g}^{(n)}=\{0\}$ and *countably nilpotent* if $\bigcap_{n=1}^{\infty}\mathfrak{g}^{[n]}=\{0\}$. If a Lie algebra $\mathfrak{g}$ has a unique largest countably solvable ideal, then it will be called the radical $\mathfrak{r}(\mathfrak{g})$, and if it has a largest countably nilpotent ideal, then it will be called the nilradical $\mathfrak{n}(\mathfrak{g})$.

If the pro-Lie algebra $\mathfrak{g}$ happens to have a unique smallest member among the family of all closed ideals $\mathfrak{i}$ such that $\mathfrak{g}/\mathfrak{i}$ is reductive, then it is called the *coreductive radical* $\mathfrak{n}_{\text{cored}}(\mathfrak{g})$. □

Theorem 4.3. *Every pro-Lie algebra* $\mathfrak{g}$ *has a closed radical, a closed nilradical and a coreductive radical such that the following properties are satisfied:*
(i) $\mathfrak{n}_{\text{cored}}(\mathfrak{g})\subseteq\mathfrak{n}(\mathfrak{g})\subseteq\mathfrak{r}(\mathfrak{g})$.
(ii) $\mathfrak{n}_{\text{cored}}(\mathfrak{g})=\overline{[\mathfrak{g},\mathfrak{g}]}\cap\mathfrak{r}(\mathfrak{g})=\overline{[\mathfrak{g},\mathfrak{r}(\mathfrak{g})]}$
(iii) $\mathfrak{g}/\mathfrak{r}(\mathfrak{g})$ *is semisimple and* $\mathfrak{g}/\mathfrak{n}_{\text{cored}}(\mathfrak{g})$ *is reductive.*

Proof. [6], 6.48, 6.66, 6.67. □

For finite dimensional Lie algebras, these are standard facts, but for pro-Lie algebras, a lot is to be proved here. Solvability for infinite dimensional Lie algebras is really a transfinite concept involving ordinals, and for topological Lie algebras we must also consider the closed commutator series. It turns out that with pro-Lie algebras one never has to go beyond the commutator sequence indexed by natural numbers, and that the algebraic and topological concepts of solvability agree. Similar comments apply to nilpotency. An effective treatment of semisimplicity and reductivity involves the duality of weakly complete topological vector spaces applied to $\mathfrak{g}$-modules.

But indeed more is true.

Definition 4.4. For a a pro-Lie algebra $\mathfrak{g}$, a subalgebra $\mathfrak{s}$ is called a *Levi summand* if the function

$$(X,Y)\mapsto X+Y:\mathfrak{r}(\mathfrak{g})\times\mathfrak{s}\to\mathfrak{g}$$

is an isomorphism of topological vector spaces. □

For each X in a pro-Lie algebra $\mathfrak{g}$, a derivation $\operatorname{ad}X$ and an automorphism of topological Lie algebras $e^{\operatorname{ad}X}$ are defined by $(\operatorname{ad}X)(Y)=[X,Y]$ and $e^{\operatorname{ad}X}(Y)=$

$\sum_{n=1}^{\infty} \frac{1}{n!} \cdot (\operatorname{ad} X)^n(Y)$, where the infinite series is summable (that is, the net of finite partial sums converges for all X and Y).

Theorem 4.5. (The Levi-Mal'cev-Theorem for Pro-Lie Algebras) *Every pro-Lie algebra $\mathfrak{g}$ has Levi summands $\mathfrak{s}$ so that $\mathfrak{g}$ is algebraically and topologically the semidirect sum $\mathfrak{r}(\mathfrak{g}) \oplus \mathfrak{s}$. Each Levi summand $\mathfrak{s} \cong \mathfrak{g}/\mathfrak{r}(\mathfrak{g})$ is semisimple. For two Levi-summands $\mathfrak{s}$ and $\mathfrak{s}_*$ there is an element $X \in \mathfrak{n}_{\text{cored}}(\mathfrak{g})$ in the coreductive radical such that $\mathfrak{s}_* = e^{\operatorname{ad} X}\mathfrak{s}$.*

Proof. [6], 6.52., 6.76. □

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