Exponentiability in categories of lax algebras

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Dedicated to Nico Pumplün on the occasion of his seventieth birthday

Abstract

For a complete cartesian-closed category $\mathbf{V}$ with coproducts, and for any pointed endofunctor T of the category of sets satisfying a suitable Beck-Chevalley-type condition, it is shown that the category of lax reflexive $(T, \mathbf{V})$-algebras is a quasitopos. This result encompasses many known and new examples of quasitopoi.

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Key words: lax algebra, partial product, locally cartesian-closed category, quasitopos.

0 Introduction

Failure to be cartesian closed is one of the main defects of the category of topological spaces. But often this defect can be side-stepped by moving temporarily into the quasitopos hull of **Top**, the category of pseudotopological (or Choquet) spaces, see for example [11, 14, 7]. A pseudotopology on a set X is most easily described by a relation $\mathfrak{x} \to x$ between ultrafilters $\mathfrak{x}$ on X and points x in X, the only requirement for which is the *reflexivity* condition $\dot{x} \to x$ for all $x \in X$, with $\dot{x}$ denoting the principal ultrafilter on x. In this setting, a topology on X is a pseudotopology which satisfies the *transitivity* condition

$$\mathfrak{X} \to \mathfrak{y} \ \& \ \mathfrak{y} \to z \quad \Rightarrow \quad m(\mathfrak{X}) \to z$$

for all $z \in X$, $\mathfrak{y} \in UX$ (the set of ultrafilters on X) and $\mathfrak{X} \in UUX$; here the relation $\to$ between UX and X has been naturally extended to a relation between UUX and UX, and $m = m_X : UUX \to UX$ is the unique map that gives U together with $e_X(x) = \dot{x}$ the structure of a monad $\mathsf{U} = (U, e, m)$. Barr [2] observed that the two conditions, reflexivity and transitivity, are precisely the two basic laws of a lax Eilenberg-Moore algebra when one extends the **Set**-monad U to a lax monad of Rel(**Set**), the category of sets with relations as morphisms. In [9] Barr's presentation of topological spaces was extended to include Lawvere's presentation of metric spaces as $\mathbf{V}$-categories with $\mathbf{V} = \overline{\mathbb{R}}_+$, the extended real half-line. Thus, for any symmetric monoidal category $\mathbf{V}$ with coproducts preserved by the tensor product, and for any **Set**-monad T that suitably extends from **Set**-maps to all $\mathbf{V}$-matrices (or "$\mathbf{V}$-relations", with

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ordinary relations appearing for $\mathbf{V} = \mathbf{2}$, the two-element chain), the paper [9] develops the notion of reflexive and transitive $(\mathsf{T}, \mathbf{V})$-algebra, investigates the resulting category $\mathrm{Alg}(\mathsf{T}, \mathbf{V})$, and presents many examples, in particular $\mathbf{Top} = \mathrm{Alg}(\mathsf{U}, \mathbf{2})$.

The purpose of this paper is to show that dropping the transitivity condition leads us to a quasitopos not only in the case of **Top**, but rather generally. In order to define just reflexive $(\mathsf{T}, \mathbf{V})$-algebras, one indeed needs neither the tensor product of $\mathbf{V}$ (just the "unit" object) nor the "multiplication" of the monad T. Positively speaking then, we start off with a category $\mathbf{V}$ with coproducts and a distinguished object I in $\mathbf{V}$ and any pointed endofunctor T of **Set** and define the category $\mathrm{Alg}(T, \mathbf{V})$. Our main result says that when $\mathbf{V}$ is complete and locally cartesian closed and a certain Beck-Chevalley condition is satisfied, also $\mathrm{Alg}(T, \mathbf{V})$ is locally cartesian closed (Theorem 2.7).

Defining reflexive $(T, \mathbf{V})$-algebras for the "truncated" data T, $\mathbf{V}$ entails a considerable departure from [9], as it is no longer possible to talk about the bicategory $\mathrm{Mat}(\mathbf{V})$ of $\mathbf{V}$-matrices. The missing tensor product prevents us from being able to introduce the (horizontal) matrix composition; however, "whiskering" by **Set**-maps (considered as 1-cells in $\mathrm{Mat}(\mathbf{V})$) is still well-defined and well-behaved, and this is all that is needed in this paper.

We explain the relevant properties of $\mathrm{Mat}(\mathbf{V})$ in Section 1 and define the needed Beck-Chevalley condition. Briefly, this condition says that the comparison map that "measures" the extent to which the T-image of a pullback diagram in **Set** still is a pullback diagram must be a lax epimorphism when considered a 1-cell in $\mathrm{Mat}(\mathbf{V})$. Having presented our main result, at the end of Section 2 we show that this condition is equivalent to asking T to preserve pullbacks *or*, if $\mathbf{V}$ is thin (i.e., a preordered class), to transform pullbacks into weak pullback diagrams (barring trivial choices for I and $\mathbf{V}$). In certain cases, (BC) turns out to be even a necessary condition for local cartesian closedness of $\mathrm{Alg}(T, \mathbf{V})$, see 2.10. In Section 3 we show how to construct limits and colimits in $\mathrm{Alg}(T, \mathbf{V})$ in general, and Section 4 presents the construction of partial map classifiers, leading us to the theorem stated in the Abstract. A list of examples follows in Section 5.

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1 V-matrices

1.1 Let $\mathbf{V}$ be a category with coproducts and a distinguished object I. A $\mathbf{V}$*-matrix* (or $\mathbf{V}$*-relation*) r from a set X to a set Y, denoted by $r : X \nrightarrow Y$, is a functor $r : X \times Y \to \mathbf{V}$, i.e. an $X \times Y$-indexed family $(r(x, y))_{x,y}$ of objects in $\mathbf{V}$. With X, Y fixed, such $\mathbf{V}$-matrices form the objects of a category $\mathrm{Mat}(\mathbf{V})(X, Y)$, the morphisms $\varphi : r \to s$ of which are natural transformations, i.e. families $(\varphi_{x,y} : r(x, y) \to s(x, y))_{x,y}$ of morphisms in $\mathbf{V}$; briefly,

$$\mathrm{Mat}(\mathbf{V})(X, Y) = \mathbf{V}^{X \times Y}.$$

1.2 Every **Set**-map $f : X \to Y$ may be considered as a **V**-matrix $f : X \nrightarrow Y$ when one puts

$$f(x,y) = \begin{cases} I & \text{if } f(x) = y, \\ 0 & \text{else,} \end{cases}$$

with 0 denoting a fixed initial object in **V**. This defines a functor

$$\mathbf{Set}(X,Y) \longrightarrow \mathrm{Mat}(\mathbf{V})(X,Y),$$

of the discrete category $\mathbf{Set}(X,Y)$, and the question is: when do we obtain a full embedding, for all X and Y? Precisely when

(*) $\mathbf{V}(I,0) = \emptyset$ and $|\mathbf{V}(I,I)| = 1$,

as one may easily check. In the context of a cartesian-closed category **V**, we usually pick for I a terminal object 1 in **V**, and then condition (*) is equivalently expressed as

(**) $0 \not\cong 1$,

preventing **V** from being equivalent to the terminal category.

1.3 While in this paper we do not need the horizontal composition of **V**-matrices in general, we do need the composites sf and gr for maps $f : X \to Y$, $g : Y \to Z$ and **V**-relations $r : X \nrightarrow Y$, $s : Y \nrightarrow Z$, defined by

$$\begin{aligned} (sf)(x,z) &= s(f(x),z), \\ (gr)(x,z) &= \sum_{y\,:\,g(y)=z} r(x,y), \end{aligned}$$

for $x \in X$, $z \in Z$; likewise for morphisms $\varphi : r \to r'$ and $\psi : s \to s'$. Hence, we have the "whiskering" functors

$$-f : \mathrm{Mat}(\mathbf{V})(Y,Z) \to \mathrm{Mat}(\mathbf{V})(X,Z),$$

$$g- : \mathrm{Mat}(\mathbf{V})(X,Y) \to \mathrm{Mat}(\mathbf{V})(X,Z).$$

The horizontal composition with **Set**-maps from either side is associative up to coherent isomorphisms whenever defined; hence, if $h : U \to X$ and $k : Z \to V$, then

$$(sf)h = s(fh) \quad \text{and} \quad k(gr) \cong (kg)r.$$

Although $\mathrm{Mat}(\mathbf{V})$ falls short of being a bicategory, even a sesquicategory [15], we refer to sets as 0-cells of $\mathrm{Mat}(\mathbf{V})$, **V**-matrices as its 1-cells, and natural transformations between them as its 2-cells.

1.4 The transpose $r^\circ : Y \nrightarrow X$ of a **V**-matrix $r : X \nrightarrow Y$ is defined by $r^\circ(y,x) = r(x,y)$ for all $x \in X$, $y \in Y$. Obviously $r^{\circ\circ} = r$, and with

$$(sf)^\circ = f^\circ s^\circ,\ (gr)^\circ = r^\circ g^\circ$$

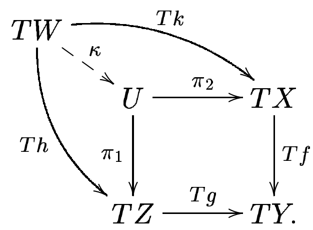
we can also introduce whiskering by transposes of **Set**-maps from either side, also for 2-cells.

A **Set**-map $f : X \to Y$ gives rise to 2-cells

$$\eta : 1_X \to f^\circ f, \;\; \varepsilon : f f^\circ \to 1_Y$$

satisfying the triangular identities $(\varepsilon f)(f\eta) = 1_f$, $(f^\circ\varepsilon)(\eta f^\circ) = 1_f$.

1.5 For a functor $T : \mathbf{Set} \to \mathbf{Set}$, we denote by $\kappa : TW \to U$ the comparison map from the T-image of the pullback $W := Z \times_Y X$ of (g, f) to the pullback $U := TZ \times_{TZ} TX$ of (Tg, Tf)



(1)

We say that the **Set**-functor T satisfies the *Beck-Chevalley Condition (BC)* if the 1-cell κ is a lax epimorphism; that is, if the "whiskering" functor $-\kappa : \mathrm{Mat}(\mathbf{V})(TW, S) \to \mathrm{Mat}(\mathbf{V})(U, S)$ is full and faithful, for every set S.

In the next section we will relate this condition with other known formulations of the Beck-Chevalley condition.

2 Local cartesian closedness of $\mathrm{Alg}(T, \mathbf{V})$

2.1 Let (T, e) be a pointed endofunctor of **Set** and **V** category with coproducts and a distinguished object I. A *lax (reflexive)* $(T, \mathbf{V})$*-algebra* (X, a, η) is given by a set X, a 1-cell $a : TX \nrightarrow X$ and a 2-cell $\eta : 1_X \to ae_X$ in $\mathrm{Mat}(\mathbf{V})$. The 2-cell η is completely determined by the **V**-morphisms

$$\eta_x := \eta_{x,x} : I \longrightarrow a(e_X(x), x),$$

$x \in X$. As we shall not change the notation for this 2-cell, *we write* (X, a) *instead of* (X, a, η). A *(lax) homomorphism* $(f, \varphi) : (X, a) \to (Y, b)$ of $(T, \mathbf{V})$-algebras is given by a map $f : X \to Y$ in **Set** and a 2-cell $\varphi : fa \to b(Tf)$ which must preserve the units: $(\varphi e_X)(f\eta) = \eta f$. The 2-cell φ is completely determined by a family of **V**-morphisms

$$f_{\mathfrak{x},x} : a(\mathfrak{x}, x) \longrightarrow b(Tf(\mathfrak{x}), f(x)),$$

$x \in X$, $\mathfrak{x} \in TX$, and preservation of units now reads as $f_{e_X(x),x}\eta_x = \eta_{f(x)}$ for all $x \in X$. For simplicity, *we write* f *instead of* (f, φ), and when we write

$$f_{\mathfrak{x},x} : a(\mathfrak{x}, x) \longrightarrow b(\mathfrak{y}, y)$$

this automatically entails $\mathfrak{y} = Tf(\mathfrak{x})$ *and* $y = f(x)$; these are the **V***-components* of the homomorphism f. Composition of (f, φ) with $(g, \psi) : (Y, b) \to (Z, c)$ is defined by

$$(g, \psi)(f, \varphi) = (gf, (\psi(Tf))(g\varphi))$$

which, in the notation used more frequently, means

$$(gf)_{\mathfrak{x},x} = (a(\mathfrak{x},x) \xrightarrow{f_{\mathfrak{x},x}} b(\mathfrak{y},y) \xrightarrow{g_{\mathfrak{y},y}} c(\mathfrak{z},z)).$$

We obtain the category $\mathrm{Alg}(T,\mathbf{V})$ (denoted by $\mathrm{Alg}(T,e;\mathbf{V})$ in [9]).

2.2 Let $\mathbf{V}$ be finitely complete. The pullback (W,d) of $f:(X,a)\to(Z,c)$ and $g:(Y,b)\to(Z,c)$ in $\mathrm{Alg}(T,\mathbf{V})$ is constructed by the pullback $W = X\times_Z Y$ in **Set** and a family of pullback diagrams in $\mathbf{V}$, as follows:

$$\begin{array}{ccc} d(\mathfrak{w},w) & \xrightarrow{f'_{\mathfrak{w},w}} & b(\mathfrak{y},y) \\ {\scriptstyle g'_{\mathfrak{w},w}}\downarrow & & \downarrow{\scriptstyle g_{\mathfrak{y},y}} \\ a(\mathfrak{x},x) & \xrightarrow{f_{\mathfrak{x},x}} & c(\mathfrak{z},z) \end{array}$$

for all $w\in W$; hence,

$$d(\mathfrak{w},w) = a(Tg'(\mathfrak{w}),g'(w)) \times_c b(Tf'(\mathfrak{w}),f'(w))$$

in $\mathbf{V}$, where $g':W\to X$ and $f':W\to Y$ are the pullback projections in **Set**. For each $w=(x,y)$ in W, we define $\eta_w := <\eta_x,\eta_y>$.

2.3 Every set X carries the *discrete* $(T,\mathbf{V})$*-structure* e_X°. In fact, the 2-cell $\eta: 1_X \to e_X^\circ e_X$ making (X,e_X°) a $(T,\mathbf{V})$-algebra is just the unit of the adjunction $e_X \dashv e_X^\circ$ in $\mathrm{Mat}(\mathbf{V})$. Now $X\mapsto (X,e_X^\circ)$ defines the left adjoint of the forgetful functor

$$\mathrm{Alg}(T,\mathbf{V}) \longrightarrow \mathbf{Set}$$

since every map $f:X\to Y$ into a $(T,\mathbf{V})$-algebra (Y,b) becomes a homomorphism $f:(X,e_X^\circ)\to(Y,b)$; indeed the needed 2-cell $fe_X^\circ \to b(Tf)$ is obtained from the unit 2-cell $\eta: 1\to be_Y$ with the adjunction $e_X\dashv e_X^\circ$: it is the mate of $f\eta: f\to be_Y f = b(Tf)e_X$. In pointwise notation, for

$$f_{\mathfrak{x},x} : e_X^\circ(\mathfrak{x},x) \longrightarrow b(\mathfrak{y},y)$$

one has $f_{\mathfrak{x},x} = 1_I$ if $e_X(x)=\mathfrak{x}$; otherwise its domain is the initial object 0 of $\mathbf{V}$, i.e. it is *trivial.*

2.4 We consider the discrete structure in particular on a one-element set 1. Then, for every $(T,\mathbf{V})$-algebra (X,a), an element $x\in X$ can be equivalently considered as a homomorphism $x:(1,e_1^\circ)\to(X,a)$ whose only non-trivial component is the unit $\eta_x: I\to a(e_X(x),x)$.

2.5 Assume $\mathbf{V}$ to be complete and locally cartesian closed. For a homomorphism $f:(X,a)\to(Y,b)$ and an additional $(T,\mathbf{V})$-algebra (Z,c) we form a substructure of the partial product of the underlying **Set**-data (see [10]), namely

$$\begin{array}{ccccc} Z & \xleftarrow{\mathrm{ev}} & Q & \xrightarrow{q} & X \\ & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ & & P & \xrightarrow{p} & Y, \end{array} \tag{2}$$

with

$$P = Z^f = \{(s,y) \mid y \in Y,\ s : (X_y, a_y) \to (Z,c)\},$$
$$Q = Z^f \times_Y X = \{(s,x) \mid x \in X,\ s : (X_{f(x)}, a_{f(x)}) \to (Z,c)\},$$

where $(X_y = f^{-1}y, a_y)$ is the domain of the pullback

$$i_y : (X_y, a_y) \longrightarrow (X, a)$$

of $y : (1, e_1^\circ) \to (Y, b)$ along f. Of course, p and q are projections, and ev is the evaluation map. We must find a structure $d : TP \nrightarrow P$ which, together with a 2-cell η, will make these maps morphisms in $\mathrm{Alg}(T, \mathbf{V})$.

For $(s,y) \in P$ and $\mathfrak{p} \in TP$, in order to define $d(\mathfrak{p}, (s,y))$, consider each pair $x \in X$ and $\mathfrak{q} \in TQ$ with $f(x) = y$ and $Tf'(\mathfrak{q}) = \mathfrak{p}$ and form the partial product

$$\begin{array}{ccccc} c(\mathfrak{z}, s(x)) & \xleftarrow{\tilde{\mathrm{ev}}_{\mathfrak{q},x}} & c(\mathfrak{z}, s(x))^{f_{\mathfrak{x},x}} \times_b a(\mathfrak{x}, x) & \longrightarrow & a(\mathfrak{x}, x) \\ & & \downarrow & & \downarrow{\scriptstyle f_{\mathfrak{x},x}} \\ & & c(\mathfrak{z}, s(x))^{f_{\mathfrak{x},x}} & \xrightarrow{\tilde{p}_{\mathfrak{q},x}} & b(\mathfrak{y}, y) \end{array} \qquad (3)$$

in $\mathbf{V}$, where $\mathfrak{z} = T\mathrm{ev}(\mathfrak{q})$, and then the multiple pullback $d(\mathfrak{p}, (s,y))$ of the morphisms $\tilde{p}_{\mathfrak{q},x}$ in $\mathbf{V}$, as in:

$$\begin{array}{ccccc} & & c(\mathfrak{z}, s(x))^{f_{\mathfrak{x},x}} & & \\ & \nearrow{\scriptstyle \pi_{\mathfrak{q},x}} & & \searrow{\scriptstyle \tilde{p}_{\mathfrak{q},x}} & \\ d(\mathfrak{p}, (s,y)) & & \xrightarrow[p_{\mathfrak{p},(s,y)}]{} & & b(\mathfrak{y}, y). \end{array}$$

2.6 We define the 2-cell $\eta : 1_P \to de_P$ componentwise. Let $(s,y) \in P$ and consider each $x \in X$ and $\mathfrak{q} \in TQ$ with $f(x) = y$ and $Tf'(\mathfrak{q}) = e_P(s,y) = T(s,y)e_1$ (where $(s,y) : 1 \to P$). Consider the pullback $j_y : X_y \to Q$ of $(s,y) : 1 \to P$ along f' in **Set**; whence, $j_y(x) = s(x)$. By (BC) there is $\mathfrak{x} \in TX_y$ such that $Tj_y(\mathfrak{x}) = \mathfrak{q}$ and $T!(\mathfrak{x}) = e_1(*)$ (where $! : X_y \to 1$ and $*$ is the only point of 1). Since $\mathrm{ev} j_y = s$, we may form the diagram

$$\begin{array}{ccccc} c(\mathfrak{z}, s(x)) & \xleftarrow{s_{\mathfrak{x},x}} & a_y(\mathfrak{x}, x) & \xrightarrow{(i_y)_{\mathfrak{x},x}} & a(\mathfrak{x}, x) \\ & & \downarrow & & \downarrow{\scriptstyle f_{\mathfrak{x},x}} \\ & & I & \xrightarrow{\eta_y} & b(e_Y(y), y) \end{array}$$

in $\mathbf{V}$, where $\mathfrak{z} = T\mathrm{ev}(\mathfrak{q}) = Ts(\mathfrak{x})$, and the square is a pullback. The universal property of (3) guarantees the existence of $\tilde{\eta}_{\mathfrak{q},x} : I \to c(\mathfrak{z}, s(x))^{f_{\mathfrak{x},x}}$ such that $\tilde{p}_{\mathfrak{q},x}\tilde{\eta}_{\mathfrak{q},x} = \eta_y$ and $\tilde{\mathrm{ev}}_{\mathfrak{q},x}(\tilde{\eta}_{\mathfrak{q},x} \times_b 1) = s_{\mathfrak{x},x}$. Then, with the multiple pullback property, the morphisms $\tilde{\eta}_{\mathfrak{q},x}$ define jointly $\eta_{(s,y)} : I \to d(e_P(s,y), (s,y))$.

2.7 Theorem. *If the pointed* **Set**-*functor T satisfies (BC) and* $\mathbf{V}$ *is complete and locally cartesian closed, then also* $\mathrm{Alg}(T, \mathbf{V})$ *is locally cartesian closed.*

Proof. Continuing in the notation of 2.5 and 2.6, we equip Q with the lax algebra structure $r : TQ \nrightarrow Q$ that makes the square of diagram (2) a pullback diagram in $\mathrm{Alg}(T, \mathbf{V})$. Then the

2-cell defined by

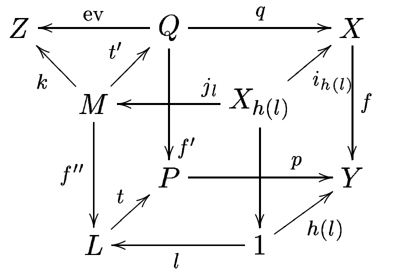
$$r(\mathfrak{q},(s,x)) \xrightarrow{\pi_{\mathfrak{q},x}\times_b 1} c(\mathfrak{z},s(x))^{f_{\mathfrak{x},x}} \times_b a(\mathfrak{x},x) \xrightarrow{\tilde{\mathrm{ev}}_{\mathfrak{q},x}} c(\mathfrak{z},s(x))$$

makes $\mathrm{ev}:(Q,r)\to(Z,c)$ a homomorphism.

In order to prove the universal property of the partial product, given any other pair $(h:(L,u)\to(Y,b), k:(M,v)\to(Z,c))$, where $M:=L\times_Y X$, we consider the map $t:L\to P$, defined by $t(l):=(s_l,h(l))$, with

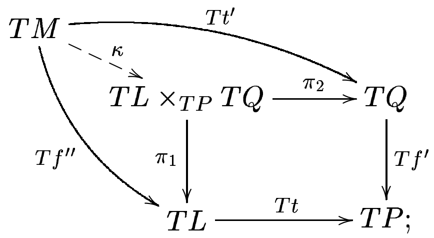
$$((X_{h(l)},a_{h(l)})) \xrightarrow{s_l} (Z,c)) = ((X_{h(l)},a_{h(l)}) \xrightarrow{j_l} (M,v) \xrightarrow{k} (Z,c)),$$

where j_l is the pullback of $l:(1,e_1^\circ)\to(L,u)$ along $f'':(M,v)\to(L,u)$. We remark that in the commutative diagram

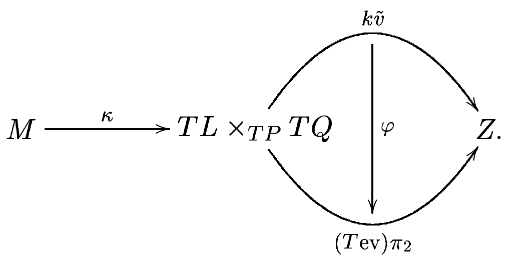


every vertical face of the cube is a pullback in **Set**.

Now, for each $l\in L$ and $\mathfrak{l}\in L$ we define $t_{\mathfrak{l},l}:u(\mathfrak{l},l)\to d(Tt(\mathfrak{l}),t(l))$ componentwise. Since $\mathrm{ev}t'=k$ we observe that Tk factors through the comparison map $\kappa:TM\to TL\times_{TP}TQ$, defined by the diagram



that is $Tk=(T\mathrm{ev})(Tt')=(T\mathrm{ev})\pi_2\kappa$. Since also kv factors through κ, i.e., $kv=k\tilde{v}\kappa$, with (BC) we conclude that the 2-cell $kv\to c(Tk)$ is of the form



For each $x\in X$ and $\mathfrak{q}\in TQ$ such that $f(x)=h(l)$ and $Tf'(\mathfrak{q})=Tt(\mathfrak{l})$, let $\mathfrak{m}\in TM$ be such that $(Tf'')(\mathfrak{m})=\mathfrak{l}$ and $(Tt')(\mathfrak{m})=\mathfrak{q}$. In the diagram

$$\begin{array}{ccccc}
c(\mathfrak{z},s_l(x)) & \xleftarrow{k_{\mathfrak{m},(l,x)}} & v(\mathfrak{m},(l,x)) & \longrightarrow & a(\mathfrak{x},x) \\
 & & \downarrow & & \downarrow {\scriptstyle f_{\mathfrak{x},x}} \\
 & & u(\mathfrak{l},l) & \xrightarrow{h_{\mathfrak{l},l}} & b(\mathfrak{y},y)
\end{array}$$

in $\mathbf{V}$ one has $\mathfrak{z} = (T\mathrm{ev})(\mathfrak{q})$ and the morphism $k_{\mathfrak{m},(l,x)}$ depends only on $\mathfrak{q}$ and $\mathfrak{l}$. Moreover, the square is a pullback, hence there is a $\mathbf{V}$-morphism $\tilde{t}_{\mathfrak{l},l} : u(\mathfrak{l}, l) \to c(\mathfrak{z}, s_l(x))^{f_{\mathfrak{x},x}}$ such that $\tilde{p}_{\mathfrak{q},x}\tilde{t}_{\mathfrak{l},l} = h_{\mathfrak{l},l}$ and $k_{\mathfrak{m},(l,x)}(\tilde{t}_{\mathfrak{l},l} \times_b 1) = \tilde{\mathrm{ev}}_{\mathfrak{q},x}$. With the multiple pullback property, the morphisms $\tilde{t}_{\mathfrak{l},l}$ define the unique 2-cell that makes $t : (L, u) \to (P, d)$ a homomorphism. $\square$

If in the proof we take for (Y, b) the terminal object of $\mathrm{Alg}(T, \mathbf{V})$, that is, the pair $(1, \top)$ where the lax structure $\top$ is constantly equal to the terminal object of $\mathbf{V}$, we conclude:

2.8 Corollary. *If the pointed* **Set***-functor T satisfies (BC) and* $\mathbf{V}$ *is complete and cartesian closed, then also* $\mathrm{Alg}(T, \mathbf{V})$ *is cartesian closed.*

We explain now the strength of our Beck-Chevalley condition.

2.9 Proposition. *For T and* $\mathbf{V}$ *as in 1.5, let* $\mathbf{V}(I, 0) = \emptyset$. *Then:*

(a) *If T satisfies (BC), then T transforms pullbacks into weak pullbacks. The two conditions are actually equivalent when* $\mathbf{V}$ *is thin (i.e. a preordered class).*

(b) *If* $\mathbf{V}$ *is not thin, satisfaction of (BC) by T is equivalent to preservation of pullbacks by T.*

(c) *If* $\mathbf{V}$ *is cartesian closed, with $I = 1$ the terminal object, then T satisfies (BC) if and only if* $(Tf)^\circ Tg = Tk(Th)^\circ$, *for every pullback diagram*

$$\begin{array}{ccc} W & \xrightarrow{k} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{g} & Y \end{array} \tag{4}$$

in **Set**.

Proof. (a) Let $\kappa : TW \to U$ be the comparison map of diagram (1). By (BC) the 2-cell $\kappa\eta : \kappa \to \kappa\kappa^\circ\kappa$ is the image by $-\kappa$ of a 2-cell $\sigma : 1_U \to \kappa\kappa^\circ$. Hence, for each $u \in U$ there is a $\mathbf{V}$-morphism $I \to \kappa\kappa^\circ(u, u) = \sum_{\mathfrak{w} \in TW\,:\,\kappa(\mathfrak{w}) = u} \kappa(\mathfrak{w}, u)$. Therefore the set $\{\mathfrak{w} \in TW \mid \kappa(\mathfrak{w}) = u\}$ cannot be empty, that is, κ is surjective.

If $\mathbf{V}$ is thin and κ is surjective, there is a (necessarily unique) 2-cell $1_U \to \kappa\kappa^\circ$. Then each 2-cell $\psi : \kappa r \to \kappa s$ induces a 2-cell $\varphi : r \to s$ defined by

$$r \xrightarrow{r\sigma} r\kappa\kappa^\circ \xrightarrow{\psi\kappa^\circ} s\kappa\kappa^\circ \xrightarrow{s\varepsilon} s$$

whose image under $-\kappa$ is necessarily ψ.

(b) If T preserves pullbacks, then κ is an isomorphism and (BC) holds.

Conversely, let T satisfy (BC) and let $\kappa : TW \to U$ be a comparison map as in (1). We consider $\mathfrak{w}_0$, $\mathfrak{w}_1 \in TW$ with $\kappa(\mathfrak{w}_0) = \kappa(\mathfrak{w}_1)$ and $\mathbf{V}$-morphisms $\alpha, \beta : v \to v'$ with $\alpha \neq \beta$, and define $r : U \times U \to \mathbf{V}$ by $r(u, u') = v$ and $s : U \times U \to \mathbf{V}$ by $s(u, u') = v'$. The 2-cell $\psi : r\kappa \to s\kappa$, with $\psi_{\mathfrak{w},u} = \alpha$ if $\mathfrak{w} = \mathfrak{w}_0$ and $\psi_{\mathfrak{w},u} = \beta$ elsewhere, factors through κ only if $\mathfrak{w}_0 = \mathfrak{w}_1$.

(c) For any commutative diagram (4) there is a 2-cell $kh^\circ \to f^\circ g$, defined by

$$kh^\circ \xrightarrow{\eta kh^\circ} f^\circ f kh^\circ = f^\circ g h h^\circ \xrightarrow{f^\circ g\varepsilon} f^\circ g,$$

which is an identity morphism in case the diagram is a pullback.

If T satisfies (BC) and $\mathbf{V}$ is not thin, the equality $Tk(Th)^\circ = (Tf)^\circ Tg$ follows from (b). If $\mathbf{V}$ is thin, then in the diagram (1) the 2-cell $\sigma : 1 \to \kappa\kappa^\circ$ considered in (a) gives rise to a 2-cell

$$(Tf)^\circ Tg = \pi_2\pi_1^\circ \xrightarrow{\pi_2\sigma\pi_1^\circ} \pi_2\kappa\kappa^\circ\pi_1^\circ = Tk(Th)^\circ,$$

and the equality follows.

Conversely, the equality $(Tf)^\circ Tg = Tk(Th)^\circ$ guarantees the surjectivity of κ, hence (BC) follows in case $\mathbf{V}$ is thin, by (a). If $\mathbf{V}$ is not thin, we first observe that a coproduct $\sum_X I$ is isomorphic to I only if X is a singleton, due to the cartesian closedness of $\mathbf{V}$. Now, $(Tf)^\circ Tg = Tk(Th)^\circ$ means that, for every $\mathfrak{z} \in TZ$ and $\mathfrak{x} \in TX$ with $Tg(\mathfrak{z}) = Tf(\mathfrak{x})$,

$$I = Tf(\mathfrak{x}, Tg(\mathfrak{z})) = Tf^\circ Tg(\mathfrak{z}, \mathfrak{x}) = TkTh^\circ(\mathfrak{z}, \mathfrak{x}) = \sum\{I \mid \mathfrak{w} \in TW \;:\; Tk(\mathfrak{w}) = \mathfrak{x} \;\&\; Th(\mathfrak{w}) = \mathfrak{z}\}.$$

From this equality we conclude that there exists exactly one such $\mathfrak{w}$, i.e. $TW = TZ \times_{TY} TX$. □

2.10 Finally we remark that, in some circumstances, the 2-categorical part of (BC) is essential for local cartesian-closedness of $\mathrm{Alg}(T, \mathbf{V})$. Indeed, if $\mathbf{V}$ is extensive [4], T transforms pullback diagrams into weak pullback diagrams and $\mathrm{Alg}(T, \mathbf{V})$ is locally cartesian closed, then T satisfies (BC), as we show next. To check (BC) we consider a 2-cell $\psi : r\kappa \to s\kappa$, with $\kappa : TW \to U$ the comparison map of diagram (1) and $r, s : U \to S$. We need to check that $\psi = \varphi\kappa$ for a unique 2-cell $\varphi : r \to s$. This 2-cell exists, and it is unique if and only if

$$\forall \mathfrak{w}_0, \mathfrak{w}_1 \in TW \;\; \forall s \in S \;\; \kappa(\mathfrak{w}_0) = \kappa(\mathfrak{w}_1) \;\;\Rightarrow\;\; \psi_{\mathfrak{w}_0, s} = \psi_{\mathfrak{w}_1, s}.$$

For $v := r(\kappa(\mathfrak{w}_0), s)$ and $v' := s(\kappa(\mathfrak{w}_0), s)$, and $\alpha := \psi_{\mathfrak{w}_0, s}$ and $\beta = \psi_{\mathfrak{w}_1, s}$, we want to show that $\alpha = \beta$.

For that, in the pullback diagram (4) we consider structures a, b, c, d, on X, Y, Z and W respectively, constantly equal to $I + v$, with $\eta : I \to I + v$ the coproduct injection. For d' constantly equal to $I + v'$, in the diagram

$$\begin{array}{ccccc} (W, d') & \xleftarrow{(\mathrm{id}, \varepsilon)} & (W, d) & \xrightarrow{(k,1)} & (X, a) \\ & & {\scriptstyle (h,1)}\downarrow & & \downarrow{\scriptstyle (f,1)} \\ & & (Z, c) & \xrightarrow{(g,1)} & (Y, b) \end{array}$$

we define ε by:

$$\varepsilon_{\mathfrak{w}, w} = \begin{cases} 1 + \alpha & \text{if } \mathfrak{w} = \mathfrak{w}_0, \\ 1 + \beta & \text{elsewhere.} \end{cases}$$

The square is a pullback. Hence the morphism $(\mathrm{id}, \varepsilon)$ factors through the partial product via $t \times_Y \mathrm{id}$, with $t : Z \to P$. Since the 2-cell of $t \times_Y \mathrm{id}$ is obtained by a pullback construction and $\kappa(\mathfrak{w}_0) = \kappa(\mathfrak{w}_1)$, its 2-cell "identifies" $\mathfrak{w}_0$ and $\mathfrak{w}_1$, hence $\varepsilon_{\mathfrak{w}_0, w} = \varepsilon_{\mathfrak{w}_1, w}$, that is, $1 + \alpha = 1 + \beta$. Therefore $\alpha = \beta$, by extensitivity of $\mathbf{V}$.

3 (Co)completeness of the category $\mathrm{Alg}(T,\mathbf{V})$

3.1 We assume $\mathbf{V}$ to be complete and cocomplete. The construction of limits in $\mathrm{Alg}(T,\mathbf{V})$ reduces to a combined construction of limits in **Set** and $\mathbf{V}$, as we show next.

The limit of a functor

$$\begin{array}{rcl} F:\mathbf{D} & \to & \mathrm{Alg}(T,\mathbf{V}) \\ D & \mapsto & (FD,a_D) \\ D\xrightarrow{f}E & \mapsto & (FD,a_D)\xrightarrow{Ff}(FE,a_E) \end{array}$$

is constructed in two steps.

First we consider the composition of F with the forgetful functor into **Set**

$$\mathbf{D}\xrightarrow{\ F\ }\mathrm{Alg}(T,\mathbf{V})\xrightarrow{\qquad}\mathbf{Set}, \tag{5}$$

and construct its limit in **Set**

$$(L\xrightarrow{\ p^D\ }FD)_{D\in\mathbf{D}}.$$

Then, we define the $(T,\mathbf{V})$-algebra structure $a:TL\nrightarrow L$, that is the map $a:TX\times X\to\mathbf{V}$, pointwise. For every $\mathfrak{l}\in TL$ and $l\in L$, we consider now the functor

$$\begin{array}{rcl} F_{\mathfrak{l},l}:\mathbf{D} & \to & \mathbf{V} \\ D & \mapsto & a_D(Tp^D(\mathfrak{l}),p^D(l)) \\ D\xrightarrow{f}E & \mapsto & a_D(Tp^D(\mathfrak{l}),p^D(l))\xrightarrow{Ff_{Tp^D(\mathfrak{l}),p^D(l)}}a_E(Tp^E(\mathfrak{l}),p^E(l)) \end{array}$$

and its limit in $\mathbf{V}$

$$(a(\mathfrak{l},l)\xrightarrow{\ p^D_{\mathfrak{l},l}\ }a_D(Tp^D(\mathfrak{l}),p^D(l)))_{D\in\mathbf{D}}.$$

This equips $p^D:(L,a)\to(FD,a_D)$ with a 2-cell $p^Da\to a_DTp^D$.

By construction

$$(L,a)\xrightarrow{\ p^D\ }(FD,a_D) \tag{6}$$

is a cone for F. To check that it is a limit, let

$$(Y,b)\xrightarrow{\ g^D\ }(FD,a_D)$$

be a cone for F. By construction of (L,p^D), there exists a map $t:Y\to L$ such that $p^Dt=g^D$ for each $D\in\mathbf{D}$. For each $\mathfrak{y}\in TY$ and $y\in Y$,

$$b(\mathfrak{y},y)\xrightarrow{\ g^D_{\mathfrak{y},y}\ }a_D(Tp^D(Tt(\mathfrak{y})),p^D(t(y)))$$

is a cone for the functor $F_{Tt(\mathfrak{y}),t(y)}$. Hence, by construction of $a(Tt(\mathfrak{y}),t(y))$, there exists a unique $\mathbf{V}$-morphism $t_{\mathfrak{y},y}$ making the diagram

$$\begin{array}{ccc} a(Tt(\mathfrak{y}),t(y)) & \xrightarrow{\ p^D_{\mathfrak{y},y}\ } & a_D(Tp^D(Tt(\mathfrak{y})),p^D(t(y))) \\ {\scriptstyle t_{\mathfrak{y},y}}\uparrow & \nearrow{\scriptstyle g^D_{\mathfrak{y},y}} & \\ b(\mathfrak{y},y) & & \end{array}$$

commutative. These $\mathbf{V}$-morphisms define pointwise the unique 2-cell $gb \to p^D a$.

For each $l \in L$, $\eta_l : I \to a(e_L(l), l)$ is the morphism induced by the cone

$$(\eta^D_{p^D(l),p^D(l)} : I \to a_D(e_{FD}(p^D(l)), p^D(l)))_{D \in \mathbf{D}}.$$

3.2 Cocompleteness. To construct the colimit of a functor $F : \mathbf{D} \to \mathrm{Alg}(T, \mathbf{V})$ we first proceed analogously to the limit construction. That is, we form the colimit in **Set**

$$(FD \xrightarrow{\ i^D\ } Q)_{D \in \mathbf{D}}$$

of the functor (5).

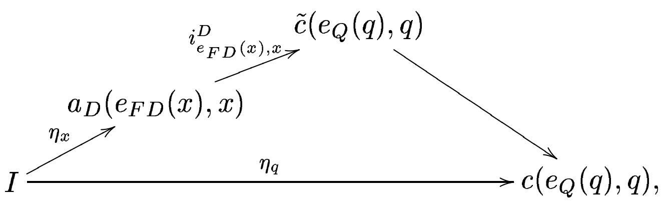
To construct the structure $c : TQ \nrightarrow Q$, for each $\mathfrak{q} \in TQ$ and $q \in Q$, we consider the functor $F^{\mathfrak{q},q} : \mathbf{D} \to \mathbf{V}$, with

$$F^{\mathfrak{q},q}(D) = \sum \{a_D(\mathfrak{x}, x) \,|\, Ti^D(\mathfrak{x}) = \mathfrak{q},\ i^D(x) = q\},$$

and, for $f : D \to E$, the morphism $F^{\mathfrak{q},q}(f) : F^{\mathfrak{q},q}(D) \to F^{\mathfrak{q},q}(E)$ is induced by

$$a_D(\mathfrak{x}, x) \xrightarrow{Ff_{\mathfrak{x},x}} a_E(Tf(\mathfrak{x}), f(x)) \longrightarrow \sum \{a_E(\mathfrak{y}, y) \,|\, Ti^E(\mathfrak{y}) = \mathfrak{q},\ i^E(y) = q\} = F^{\mathfrak{q},q}(E).$$

and denote by $\tilde{c}(\mathfrak{q}, q)$ the colimit of $F^{\mathfrak{q},q}$. If $\mathfrak{q} \neq e_Q(q)$ for $q \in Q$, then $\tilde{c}(\mathfrak{q}, q)$ is in fact the structure $c(\mathfrak{q}, q)$ on the colimit. For $\mathfrak{q} = e_Q(q)$, the multiple pushout



defines $c(e_Q(q), q)$, with $D \in \mathbf{D}$ and $x \in FD$ such that $i^D(x) = q$.

4 Representability of partial morphisms

4.1 Let $\mathcal{S}$ be a pullback-stable class of morphisms of a category $\mathbf{C}$. An $\mathcal{S}$-*partial map from* X *to* Y is a pair $(\ X \xleftarrow{\ s\ } U \longrightarrow Y\ )$ where $s \in \mathcal{S}$. We say that $\mathcal{S}$ *has a classifier* if there is a morphism true $: 1 \to \tilde{1}$ in $\mathcal{S}$ such that every morphism in $\mathcal{S}$ is, in a unique way, a pullback of true; $\mathbf{C}$ *has* $\mathcal{S}$-*partial map classifiers* if, for every $Y \in \mathbf{C}$, there is a morphism $\mathrm{true}_Y : Y \to \tilde{Y}$ in $\mathcal{S}$ such that every $\mathcal{S}$-partial map $(\ X \xleftarrow{\ s\ } U \longrightarrow Y\ )$ from X to Y can be uniquely completed so that the diagram

$$\begin{array}{ccc} U & \longrightarrow & Y \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle \mathrm{true}_Y} \\ X & \dashrightarrow & \tilde{Y}. \end{array}$$

is a pullback.

¿From Corollary 4.6 of [10] it follows that:

4.2 Proposition. *If $\mathcal{S}$ is a pullback-stable class of morphisms in a finitely complete locally cartesian-closed category* $\mathbf{C}$*, then the following assertions are equivalent:*

(i) *$\mathcal{S}$ has a classifier;*

(ii) $\mathbf{C}$ *has $\mathcal{S}$-partial map classifiers.*

4.3 Our goal is to investigate whether the category $\mathrm{Alg}(T, \mathbf{V})$ has $\mathcal{S}$-partial map classifiers, for the class $\mathcal{S}$ of extremal monomorphisms. For that we first observe:

4.4 Lemma. *An $\mathrm{Alg}(T, \mathbf{V})$-morphism $s : (U, c) \to (X, a)$ is an extremal monomorphism if and only if the map $s : U \to X$ is injective and, for each $\mathfrak{u} \in TU$ and $u \in U$, $s_{\mathfrak{u},u} : c(\mathfrak{u}, u) \to a(\mathfrak{x}, x)$ is an isomorphism in* $\mathbf{V}$.

4.5 Proposition. *In $\mathrm{Alg}(T, \mathbf{V})$ the class of extremal monomorphisms has a classifier.*

Proof. For $\tilde{1} = (1 + 1, \tilde{\top})$, where $\tilde{\top}$ is pointwise terminal, we consider the inclusion true $: 1 \to \tilde{1}$ onto the first summand. For every extremal monomorphism $s : (U, c) \to (X, a)$, we define $\chi_U : (X, a) \to \tilde{1}$ with $\chi_U : X \to 1 + 1$ the characteristic map of $s(U)$, and the 2-cell constantly $! : a(\mathfrak{x}, x) \to 1$. Then the diagram below

$$\begin{array}{ccc} (U,s) & \xrightarrow{\;!\;} & 1 \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle \mathrm{true}} \\ (X,a) & \xrightarrow{\;\chi_U\;} & \tilde{1}. \end{array}$$

is a pullback diagram; it is in fact the unique possible diagram that presents s as a pullback of true. □

Using Theorem 2.7 and Proposition 4.5, we conclude that:

4.6 Theorem. *If the pointed* **Set***-functor T satisfies (BC) and* $\mathbf{V}$ *is a complete and cocomplete locally cartesian closed category, then $\mathrm{Alg}(T, \mathbf{V})$ is a quasitopos.*

4.7 Remark. Representability of (extremal mono)-partial maps can also be proved directly, and in this way one obtains a slight improvement of Theorem 4.6: $\mathrm{Alg}(T, \mathbf{V})$ is a quasi-topos whenever T satisfies (BC) and $\mathbf{V}$ is a complete and cocomplete cartesian closed category, not necessarily locally so.

5 Examples.

5.1 We start off with the trivial functor T which maps every set to a terminal object 1 of **Set**. T preserves pullbacks. Choosing for I the top element of any (complete) lattice $\mathbf{V}$ we obtain with $\mathrm{Alg}(T, \mathbf{V})$ nothing but the topos **Set**. This shows that local cartesian closedness of $\mathbf{V}$ is not a necessary condition for local cartesian closedness of $\mathrm{Alg}(T, \mathbf{V})$. We also note that T does not carry the structure of a monad.

If, for the same T, we choose $\mathbf{V} = \mathbf{Set}$, then $\mathrm{Alg}(T, \mathbf{Set})$ is the formal coproduct completion of the category $\mathbf{Set}_*$ of pointed sets, i.e. $\mathrm{Alg}(T, \mathbf{Set}) \cong \mathrm{Fam}(\mathbf{Set}_*)$.

5.2 Let $T = \mathrm{Id}$, $e = \mathrm{id}$. Considering for $\mathbf{V}$ as in [9] the two-element chain $\mathbf{2}$, the extended half-line $\overline{\mathbb{R}}_+ = [0, \infty]$ (with the natural order reversed), and the category $\mathbf{Set}$, one obtains with $\mathrm{Alg}(T, \mathbf{V})$ the category of

- sets with a reflexive relation
- sets with a fuzzy reflexive relation
- reflexive directed graphs,

respectively.

More generally, if we let $TX = X^n$ for a non-negative integer n, with the same choices for $\mathbf{V}$ one obtains

- sets with a reflexive $(n+1)$-ary relation
- sets with a fuzzy reflexive $(n+1)$-ary relation
- reflexive directed "multigraphs" given by sets of vertices and of edges, with an edge having an ordered n-tuple of vertices as its source and a single edge as its target; reflexivity means that there is a distinguished edge $(x, \cdots, x) \to x$ for each vertex x.

Note that the case $n = 0$ encompasses Example 5.1.

5.3 For a fixed monoid M, let T belong to the monad T arising from the adjunction

$$\mathbf{Set}^M \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \mathbf{Set},$$

i.e. $TX = M \times X$ with $e_X(x) = (0, x)$, with 0 neutral in M (writing the composition in M additively). T preserves pullbacks. The quasitopos $\mathrm{Alg}(T, \mathbf{Set})$ may be described as follows. Its objects are "M-normed reflexive graphs", given by a set X of vertices and sets $a(x, y)$ of edges from x to y which come with a "norm" $v_{x,y} : a(x, y) \to M$ for all $x, y \in X$; there is a distinguished edge $1_x : x \to x$ with $v_{x,x}(1_x) = 0$. Morphisms must preserve the norm. Of course, for trivial M we are back to directed graphs as in 5.2.

It is interesting to note that if one forms $\mathrm{Alg}(\mathsf{T}, \mathbf{Set})$ for the (untruncted) monad T (see [9]), then $\mathrm{Alg}(\mathsf{T}, \mathbf{Set})$ is precisely the comma category $\mathbf{Cat}/M$, where M is considered a one-object category; its objects are categories which come with a norm function v for morphisms satisfying $v(gf) = v(g) + v(f)$ for composable morphisms f, g.

5.4 Let $T = U$ be the ultrafilter functor, as mentioned in the Introduction. U transforms pullbacks into weak pullback diagrams. Hence, for $\mathbf{V} = \mathbf{2}$ we obtain with $\mathrm{Alg}(T, \mathbf{2})$ the quasitopos of pseudotopological spaces, and for $\mathbf{V} = \overline{\mathbb{R}}_+$ the quasitopos of (what should be called) quasi-approach spaces (see [9, 8]). If we choose for $\mathbf{V}$ the extensive category $\mathbf{Set}$, then the resulting category Alg(U, **Set**) is a rather naturally defined supercategory of the category of ultracategories (as defined in [9]) but fails to be locally cartesian closed, according to 2.9(b) and 2.10.

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