The Invariant Theory of Unipotent Groups

Frank Grosshans

Aachen RWTH

June, 2010

§1. Introduction

- Notation
  - $\mathbb{C}$: complex numbers
  - V: finite-dimensional vector space over $\mathbb{C}$
  - $\mathbb{C}[V]$: polynomial functions on vector space V
  - $G \subset GL(V)$
  - the *orbit* of an element $v \in V$ is $Gv = \{g \cdot v : g \in G\}$
  - the *isotropy subgroup* of v is $G_v = \{g \in G : g \cdot v = v\}$
  - invariant polynomials: $\mathbb{C}[V]^G = \{f \in \mathbb{C}[V] : f(g \cdot v) = f(v)$ for all $g \in G, v \in V\}$
  - *unipotent algebraic group* $U \subset GL(V)$:
    conjugate to subgroup of upper triangular matrices, 1's on the diagonal.

§1. Introduction

- Questions
  - What is structure of algebra of invariants?
  - Can the algebra of invariants be used to separate orbits?
  - Can generators of the algebra of invariants be written down explicitly?

§1. Introduction

- Binary forms [8] , the groups
  - $SL_2 = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1\}$
  - $U = \{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\}$
  - $T = \{\begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}\}$

§1. Introduction

- V_d: binary forms of degree d
  - $f = a_0x^d + \binom{d}{1} a_1x^{d-1}y + \ldots + \binom{d}{i} a_ix^{d-i}y^i + \ldots + \binom{d}{d} a_dy^d$.
  - $g = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\}$ acts on V_d: $x \to (dx - by)$, $y \to (-cx + ay)$.

- Protomorphs for binary forms
  - $V_d^o = \{f \in V_d : a_1 = 0\}, V_d^{'} = \{f \in V_d : a_0 \neq 0\}$
  - Have isomorphism $\varphi : U \times V_d^o \to V_d^{'}$, $(u, v) \to u{\cdot}v$

Tan Algorithm

- Algorithm [18; p. 566] for finding $\mathbb{C}[V]^U$ (so get $\mathbb{C}[V]^G$, too)
  - Choose ℓ invariants, say, $F_1 = a_0, F_2, \ldots, F_\ell$, so that $\mathbb{C}[F_1, \ldots, F_\ell] \subset \mathbb{C}[V]^U \subset \mathbb{C}[F_1, \ldots, F_\ell][\frac{1}{a_0}]$.
  - Put $\overline{F_i} = F_i$ mod the ideal $a_0\mathbb{C}[V]$).
  - Find (finite) set of generators, say $\{p_1, \ldots, p_r\}$ for relations among $\overline{F_i}$. Then, $p_i(F_1, \ldots, F_\ell) = a_0^{s_i} f_i$.
  - Replace $\{F_1, \ldots, F_\ell\}$ by $\{F_1, \ldots, F_\ell, f_1, \ldots, f_r\}$ and repeat.

§1. Introduction

- Example
  - Binary cubics

§2. Structure of algebra of invariants

- A. Finite generation
  - Definition. k : algebraically closed field, A commutative k-algebra, G linear algebraic group with identity e. A *rational action* of G on A is given by a mapping $G \times A \longrightarrow A$, denoted by $(g, a) \longrightarrow ga$ so that: (i) $g(g'a) = (gg')a$ and $ea = a$ for all $g, g' \in G, a \in A$; (ii) the mapping $a \longrightarrow ga$ is a k-algebra automorphism for all $g \in G$; (iii) every element in A is contained in a finite-dimensional subspace of A which is invariant under G and on which G acts by a rational representation.
  - G acts *rationally* on affine variety X means G acts rationally on $k[X]$, algebra of polynomial functions on X.

§2. Structure of algebra of invariants

- A. Finite generation
  - **Theorem 1** (Weyl [20], Schiffer, Chevalley, Nagata [13], Haboush, Borel, Popov [15]; also [14]). k, algebraically closed field. Let G be a linear algebraic group. Then the following statements are equivalent: (i) G is reductive; (ii) for each finitely generated, commutative, rational G - algebra A, the algebra of invariants A^G is finitely generated over k.
  - Note: When G is reductive, minimal number of generators can be huge. Kac [11] showed that for the action of SL_2 on binary forms of odd degree d, the minimal number of generators is $\geq p(d-2)$ where p is the partition function. For upper bound on degree see [16].

§2. Structure of algebra of invariants

- A. Finite generation: localization
  - **Theorem 2** [most recent reference: 5] G linear algebraic group, X irreducible affine variety, G acts rationally on X. There is an element $a \in \mathbb{C}[X]^G$ so that $\mathbb{C}[X]^G[1/a]$ is a finitely generated $\mathbb{C}$ - algebra. The set of all such a forms a radical ideal.
  - Tan algorithm works and terminates if and only if $\mathbb{C}[X]^G$ is finitely generated $\mathbb{C}$-algebra.
  - **Theorem 3** [5] Let X be an irreducible, affine variety and let G be a unipotent linear algebraic group which acts regularly on X. Let Z be the closed set consisting of the zeros of the finite generation ideal. Then, each component of Z has codimension ≥ 2 in X.
  - **Example**: Nagata [6, p.339 and 17]

§2. Structure of algebra of invariants

- A. Finite generation: homogeneous spaces
  - **Definition.** Let G be a linear algebraic group and let H be a closed subgroup of G. Let $\mathbb{C}[G]^H = \{f \in \mathbb{C}[G] : f(gh) = f(g)$ for all $g \in G, h \in H\}$. $\mathbb{C}[G]^H = \mathbb{C}[G/H]$.
  - **Theorem 4** [9; p. 20]. Suppose that G/H is quasi-affine. Then $\mathbb{C}[G/H]$ is finitely generated if and only if there is an embedding $G/H \hookrightarrow X$, where X is an affine variety so that $\operatorname{codim}(X \backslash G/H) \geq 2$.
  - **Examples**: maximal unipotent subgroups, unipotent radicals of parabolic subgroups

§2. Structure of algebra of invariants

- A. Finite generation: homogeneous spaces
  - **Theorem 5,** the boundary ideal. [1; p.4372]. Consider an open embedding $G/H \hookrightarrow \widetilde{X}$ into affine variety $\widetilde{X}$. Let $\mathcal{I}(G/H)$ be the the radical of the ideal in $\mathbb{C}[G]^H$ generated by $\{f \in \mathbb{C}[\widetilde{X}] : f = 0 \text{ on } \widetilde{X} \backslash G/H\}$. This ideal does not depend on $\widetilde{X}$. It is smallest nonzero radical G_ℓ-invariant ideal of $\mathbb{C}[G]^H$. Also, G/H affine if and only if $\mathcal{I}(G/H) = \mathbb{C}[G]^H$.

§2. Structure of algebra of invariants

- A. Finite generation: homogeneous spaces
  - **Popov - Pommerening conjecture**: G reductive with maximal torus T, U unipotent subgroup of G normalized by T. Then $\mathbb{C}[G]^U$ is a finitely generated $\mathbb{C}$ - algebra.

§2. Structure of algebra of invariants

- A. Finite generation: homogeneous spaces
  - **Definition**. G reductive algebraic group, H a closed subgroup. Say H is an *epimorphic subgroup* of G if $\mathbb{C}[G]^H = \mathbb{C}$.
  - (F) for any finite-dimensional H-module E, the vector space $ind_H^G E = (\mathbb{C}[G] \otimes E)^H$ is finite-dimensional over $\mathbb{C}$.
  - (FG) there is a character $\chi \in X(H)$ such that the subgroup $H_\chi = \{h \in H : \chi(h) = 1\}$ satisfies: $\mathbb{C}[G]^{H_\chi}$ is a finitely generated $\mathbb{C}$-algebra.
  - (SFG) The algebra is $\mathbb{C}[G]^{\mathcal{R}_u H}$ is finitely generated over $\mathbb{C}$ where $\mathcal{R}_u H$ is unipotent radical of H. Popov-Pommerening conjecture $\Rightarrow$(SFG)
  - (SFG) $\Rightarrow$ (FG) $\Rightarrow$ (F). Nagata: (F) does not imply (FG).
  - Borel-Bien-Kollar [2]: G reductive. If H is epimorphic in G and normalized by a maximal torus, then (F).

- B. Transfer Principle
    - **Transfer Principle** [Roberts (1861), [8], also 9; p. 49]. G linear algebraic group, H a closed subgroup. Let M be a rational G - module. Then $(M \otimes \mathbb{C}[G]^H)^G \simeq M^H$ where G acts by left translation on $\mathbb{C}[G]$.
    - **Corollary**. Suppose that G is reductive and that X is an affine variety on which G acts regularly. Let $H \subset G$. If $\mathbb{C}[G]^H$ is a finitely generated $\mathbb{C}$ - algebra, then so is $\mathbb{C}[X]^H$.
    - **Example:** Weitzenböck's theorem [19]. $G/U \hookrightarrow \mathbb{A}^2$.

§3. Quotient spaces and separated orbits

- A. Rosenlicht's theorem
  - **Definition.** Let X be an irreducible algebraic variety, H an algebraic group which acts regularly on Y. A *geometric quotient* of Y by H is a pair (Y, π) where Y is an algebraic variety and $\pi : X \to Y$ is a morphism such that (i) π is open, constant on H-orbits and defines a bijection between the orbits of H and the points of Y; (ii) if $\mathcal{O}$ is an open subset of Y, the mapping $\pi^*{:}\mathbb{C}[\mathcal{O}] \to \mathbb{C}[\pi^{-1}(\mathcal{O})]^H$, given by $\pi^*(f)(x) = f(\pi(x))$, is an isomorphism.
  - **Theorem 6** (Rosenlicht) [4; p.108]: Let H be an algebraic group which operates rationally on an irreducible (algebraic) variety X. There is a non-empty, H-invariant, open set $X_o \subset X$ with a geometric quotient $\pi : X_o \to Y_o$.

- B. Separated orbits
  - **Definition** [6; p. 331]. Let X be an affine variety and let H be an algebraic group which acts regularly on X. An orbit Hx is called H-*separated* if for any $y \in X$, $y \notin Hx$, there is an $f \in \mathbb{C}[X]^H$ so that $f(y) \neq f(x)$. Let $\Omega_2(X, H)$ be the interior of the union of all the H-separated orbits.
  - Examples: GL_2 acts on $\mathbb{C}^2$; GL_n acts on $M_{n,n}$ by conjugation.

- B. Separated orbits and quotient spaces
  - **Theorem 7** [6; p. 332]**.** Let X be a quasi-affine variety and let H be an algebraic group which acts regularly on X. The variety $\Omega_2(X, H)/H$ exists, is quasi-affine, and open in the scheme $\mathrm{Spec}(\mathbb{C}[X]^H)$.
  - **Theorem 8** [6; p. 338]**.** Suppose that U is a unipotent algebraic group which acts regularly on X. Then $\Omega_2(X, U)$ is dense in X.

- C. Reductive groups
  - **Definition.** Suppose that $G \subset GL(V)$ is reductive. A point $v \in V$ is said to be *stable* if G_v is finite and Gv is closed
  - **Theorem 9** (Mumford) [4; p.138, also 14]**.** G connected, reductive, acts on an affine variety $X \subset V$

(a) A point $v \in V$ is not stable if and only if there is a multiplicative, one-parameter subgroup $\{\gamma(a) : a \in \mathbb{C}^*\}$ in G so that $\lim_{a->0} \gamma(a)v$ exists.

(b) Let X_o^S be all the stable points in X. The *geometric quotient* of X_o^S exists and is quasi-affine.

- C. Reductive groups
  - **Theorem 10.** G connected, reductive, acts on an affine variety X. The orbit Gx is separated on X if and only if it is closed in X and is of maximal dimension. (so, stable $\Rightarrow$ separated)
  - Example: binary forms

§4. Separated orbits for unipotent groups

- Program [7; p. 63 and 72]
  - U unipotent, good generalization would:
  - (1) use $\mathbb{C}[X]^U$ to separate as many orbits as possible;
  - (2) have suitable notion of stable point;
  - (3) connect (2) to creation of geometric quotient.

§4. Separated orbits for unipotent groups

- Program [7; p. 63 and 72]
  - From now on, suppose that G is semisimple and that $U \subset G$ is a unipotent subgroup. Suppose that U acts on an affine variety X. Idea is to extend this to an action of G on X (or some variety $Y \supset X$), then use theory of reductive groups to get information. Will discuss easiest case below.

§4. Separated orbits for unipotent groups

- Homogeneous spaces
  - **Theorem 11** [9] Suppose that $\mathbb{C}[G]^U$ is a finitely generated $\mathbb{C}$ - algebra. Let Z be the (normal) affine variety Z so that $\mathbb{C}[Z] = \mathbb{C}[G]^U$. There is a point $z \in Z$ so that:
  - (1) $U = G_z = \{g \in G : g{\cdot}z = z\}$;
  - (2) Z is the closure of the orbit Gz;
  - (3) G/U is isomorphic to Gz;
  - (4) $\dim(Z - Gz) \leq \dim Z - 2$.
  - Example: maximal unipotent subgroups; unipotent radicals of parabolic subgroups

- Separated orbits again
  - **Definition.** Suppose that $\mathbb{C}[G]^U$ is a finitely generated $\mathbb{C}$ - algebra and let $z \in Z$ be as above. Let G act on an affine variety X. Consider the two conditions:
  - (C1) The orbit Ux is U-separated on X.
  - (C2) The orbit $G(z, x)$ is G-separated on $Z \times X$.
  - Have (C2) $\Rightarrow$ (C1) always, but not conversely.
  - U unipotent, have (C2) $\Leftrightarrow$ (C3): (z, x) is G-stable on $Z \times X$.

§4. Separated orbits for unipotent groups

- G separated and U separated
  - **Theorem 12**.[10] Suppose that G acts on a vector space V.
  - (a) If G_v is finite, have (C1) $\Leftrightarrow$ (C2) at v.
  - (b) If $\dim V > \dim U[1 + CardW(G, T)]$, then (C1) $\Leftrightarrow$ (C2) for all $v \in V$.
  - Example: binary forms, for cubics, inequality not true but (C1) $\Leftrightarrow$ (C2) for all $v \in V_3$.

§4. Separated orbits for unipotent groups

- A quotient variety
  - **Theorem 13** [12; p.326]**.** Let $G \times_U X$ be the quotient of $G \times X$ by the free action of U defined by $u(g, x) = (gu^{-1}, ux)$.
  - (a) This quotient is a quasi-projective variety.
  - (b) If the action of U on X extends to an action of G on X, this variety is isomorphic to $(G/U) \times X$.
  - (c) $\mathbb{C}[G \times_U X]^G = \mathbb{C}[(G/U) \times X]^G = \mathbb{C}[X]^U$.
  - Example: binary forms, for cubics, inequality not true but (C1) $\Leftrightarrow$ (C2) for all $v \in V_3$.

- Separated orbits
    - **Theorem 14.** Let G be a connected semisimple algebraic group and let U be a unipotent subgroup of G. Let X be an affine variety on which G acts regularly. Suppose that (C1) $\Leftrightarrow$ (C2) for all $x \in X$. Let $X(U)$ be the set of all U-separated orbits in X. Then $X(U)$ is open, dense in X, the geometric quotient $X(U)/U$ exists, is quasi-affine, and open in the affine variety Spec $\mathbb{C}[X]^U$.

- Doran-Kirwan theory [3, p.95]
  - **Definition.** Let H act freely on an algebraic variety X and suppose that $\pi : X \to Y$ is a geometric quotient. let $x \in X$. We say that π is locally trivial at x if there is an open set $\mathcal{O} \subset Y$, and a mapping $\sigma : \mathcal{O} \to X$ so that $x \in \sigma(\mathcal{O})$, $\pi \circ \sigma = Id$, and the mapping $\tau : H \times \mathcal{O} \to V$, $(h, y) \to h{\cdot}\sigma(y)$ is an isomorphism.
  - **Theorem 15**. Let G be a connected semisimple algebraic group and let U be a unipotent subgroup of G such that $\mathbb{C}[G]^U$ is a finitely generated $\mathbb{C}$ - algebra. Let X be a normal affine variety on which G acts regularly. Suppose that (C1) $\Leftrightarrow$ (C2) for all $x \in X$. Let $X(U)$ be the set of all U-separated orbits in X and let $\pi : X(U) \to X(U)/U$ be the quotient map. Then π is locally trivial.

§4. Sources of information

- Sources
  - 1. Arzhantsev, Ivan V.: Invariant ideals and Matsushima's criterion. Comm. Algebra 36 (2008), no. 12, 4368–4374.

2. Bien, Frédéric; Borel, Armand; Kollár, János Rationally connected homogeneous spaces. Invent. Math. 124 (1996), no. 1-3, 103–127.
3. Borel, Armand Linear algebraic groups. Second edition. Graduate Texts in Mathematics, 126. Springer-Verlag, New York, 1991.
4. Brion, M.: Théorie des invariants & Géométrie des variétés quotients Hermann, Paris Éditeurs Des Sciences Et Des Arts 2000
5. Derksen, Harm; Kemper, Gregor: Computing invariants of algebraic groups in arbitrary characteristic. Adv. Math. 217 (2008), no. 5, 2089–2129.
6. Dixmier, J.; Raynaud, M.: Sur le quotient d'une variété algébrique par un groupe algébrique. Mathematical analysis and applications, Part A, pp. 327–344, Adv. in Math. Suppl. Stud., 7a, Academic Press, New York-London, 1981
7. Doran, Brent; Kirwan, Frances Towards non-reductive geometric invariant theory. Pure Appl. Math. Q. 3 (2007), no. 1, part 3, 61–105.
8. Elliott, E.B.: An Introduction to the Algebra of Binary Quantics. Second

§4. Sources of information

- Sources
  - 9. Grosshans, Frank D.: Algebraic homogeneous spaces and invariant theory. Lecture Notes in Mathematics, 1673. Springer-Verlag, Berlin, 1997.

10. Grosshans, Frank D.: Separated orbits for certain nonreductive subgroups. Manuscripta Math. 62 (1988), no. 2, 205–217.
11. Kac, Victor G.: Root systems, representations of quivers and invariant theory. Invariant theory (Montecatini, 1982), 74–108, Lecture Notes in Math., 996, Springer, Berlin, 1983.
12. Kirwan, Frances Quotients by non-reductive algebraic group actions. Moduli spaces and vector bundles, 311–366, London Math. Soc. Lecture Note Ser., 359, Cambridge Univ. Press, Cambridge, 2009

§4. Sources of information

- Sources
  - 13. Nagata, M.: Lectures on the fourteenth problem of Hilbert. Tata Institute of Fundamental Research, Bombay 1965.
  - 14. Newstead, P.E.: Newstead, P. E. Introduction to moduli problems and orbit spaces. Tata Institute of Fundamental Research Lectures on Mathematics and Physics, 51. Tata Institute of Fundamental Research, Bombay; by the Narosa Publishing House, New Delhi, 1978.

§4. Sources of information

- Sources
  - 15. Popov, V. L.: On Hilbert's theorem on invariants. Dokl. Akad. Nauk SSSR 249 (1979), no. 3, 551–555.

16. Popov, V. L.: The constructive theory of invariants. Izv. Akad. Nauk SSSR Ser. Mat. 45 (1981), no. 5, 1100–1120, 1199.
17. Steinberg, Robert: Nagata's example. Algebraic groups and Lie groups, 375–384, Austral. Math. Soc. Lect. Ser., 9, Cambridge Univ. Press, Cambridge, 1997.
18. Tan, Lin: An algorithm for explicit generators of the invariants of the basic G_a-actions. Comm. Algebra 17 (1989), no. 3, 565–572.
19. Tyc, Andrzej: An elementary proof of the Weitzenböck theorem. Colloq. Math. 78 (1998), no. 1, 123–132.
20 H. Weyl, The Classical Groups. Princeton University Press, Princeton, NJ, 1946.